

Non-Perturbative Functional Renormalization Group for Random Field Models and Related Disordered Systems. I: Effective Average Action Formalism

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We have developed a nonperturbative functional renormalization group approach for random field models and related disordered systems for which, due to the existence of many metastable states, conventional perturbation theory often fails. The approach combines an exact renormalization group equation for the effective average action with a nonperturbative approximation scheme based on a description of the probability distribution of the renormalized disorder through its cumulants. For the random field $O(N)$ model, the minimal truncation within this scheme is shown to reproduce the known perturbative results in the appropriate limits, near the upper and lower critical dimensions and at large number N of components, while providing a unified nonperturbative description of the full (N, d) plane, where d is the spatial dimension.

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I. INTRODUCTION

The effect of quenched disorder on the long-distance physics of many-body systems largely remains an unsettled question despite decades of intensive research. Ongoing controversies persist for instance on the equilibrium and out-of-equilibrium behavior of spin glasses and systems coupled to a random field.^{1,2} Even though progress has been made, it has so far proven difficult to construct a proper renormalization group (RG) approach providing a description of ordering transitions and criticality in these systems. A technical reason for this unsatisfactory situation is that quenched disorder makes the system intrinsically inhomogeneous and that one should in principle follow the renormalization of the whole probability distribution of the disorder. A physical reason is that the presence of disorder and of the resulting spatial inhomogeneity lead, for at least some range of the control parameters, to multiple “metastable states”. (At this point we use the term “metastable state” in a loose acceptance to describe configurations that minimize some energy or free-energy, action or effective action in field-theoretical terminology, but differ from the true ground state.) How such metastable states evolve upon coarse-graining under RG then represents the central issue: at large length-scale, their influence could vanish, leaving only benign signatures in the thermodynamics, or else it could modify the critical behavior of the system, the nature of its phases, and, often in an even more spectacular way, the relaxation and out-of-equilibrium dynamical properties.

A well known example of the kind of puzzles associated with quenched disorder and metastable states is the failure of the so-called “dimensional reduction” property in the random field Ising model (RFIM).^{3,4,5,6} Standard perturbation theory predicts to all orders that the critical behavior of the RFIM in dimension d is the same as that of the pure Ising model, *i.e.*, in the absence of random field, in two dimensions less, $d - 2$. The property has been shown in a compact and elegant manner by Parisi

and Sourlas⁷ by means of a supersymmetric formalism. However, dimensional reduction predicts a lower critical dimension for ferromagnetism in the RFIM of $d_{lc} = 3$, in contradiction with rigorous results.^{8,9} The dimensional reduction property must therefore break down in low enough dimension. The supersymmetric approach gives a hint at the origin of the breakdown, which appears to be related, yet in a somewhat obscure way, to the presence of multiple metastable states¹⁰ (in this case, local minima of the Hamiltonian).

Over the years, and on top of numerous computer simulations and scarce exact analytical results, theoretical approaches have been devised to cope with disordered systems characterized by multiple metastable states, such as spin glasses and random field models.¹ To list the main ones, we mention:

(i) phenomenological approaches such as the heuristic domain-wall arguments^{11,12} and the “droplet” description,^{13,14,15} in which one directly focuses on rare excitations and the associated low-energy metastable states;

(ii) mean-field theories, combined with the replica formalism in order to handle the average over disorder; for models with spin-glass ordering, the potentially dramatic effect of the metastable states is captured through a spontaneous breaking of the replica symmetry;^{1,2,16,17}

(iii) specific RG techniques for low-dimensional ($d = 1, 2$) systems, as for instance the Coulomb gas RG approach for two-dimensional disordered XY models^{18,19} or real space RG for strongly disordered one-dimensional systems;^{20,21,22}

(iv) the perturbative functional RG for energy-dominated disordered models considered in the vicinity of a critical dimension at which the fundamental fields are dimensionless;^{23,24,25,26,27,28} one must then follow the flow of a whole function, an appropriate renormalized cumulant of the disorder. As shown first by Fisher²⁸ for an elastic manifold pinned by a random potential, the long-distance physics is controlled by a zero-temperature

fixed point at which the renormalized cumulant is a non-analytic function of the fields, with the nonanalyticity encoding the effect of the many metastable states at zero temperature.

All these approaches, however, are either questionable or not easily generalizable: on the one hand, the phenomenological approaches lack rigorous foundations and the relevance of mean-field descriptions to finite-dimensional systems is, to say the least, far from guaranteed; on the other hand, the perturbative functional RG becomes extremely complex, and soon untractable in practice for random field systems when going beyond one-loop calculations;^{29,30,31} moreover, it does not allow one to study the RFIM (as for specific RG techniques, they are not extendable by construction).

The purpose of the present work, described here and in a companion paper³² is therefore to propose a general theoretical framework that leads to a consistent description of the equilibrium behavior of the random field models and related disordered systems. To achieve this, we rely on a version of Wilson's continuous RG via momentum shell integration.³³ Under various terminologies, "Exact RG", "Functional RG", and "Nonperturbative RG", it has been developed in the past 15 years to become a powerful method for investigating both universal and nonuniversal properties in Statistical Physics and Quantum Field Theory.^{34,35,36,37,38} The approach is "exact" in the sense that the RG flow associated with the progressive account of the field fluctuations over larger and larger lengthscales is described through an exact functional differential equation. It is "functional" because through the exact equation, one follows the flow of an infinite hierarchy of functions of the fields in place of simply coupling constants. It is "nonperturbative" (beyond the mere tautology that an exact description automatically includes all perturbative as well as nonperturbative effects) because it lends itself to efficient approximation schemes that are able to capture genuine nonperturbative phenomena:³⁶ to name a few, in the case of the standard $O(N)$ scalar model, (numerically) tractable approximations describe the Kosterlitz-Thouless transition of the XY model in $d = 2$, known to be associated with the binding/unbinding of topological defects (vortices), as well as the convexity property of the thermodynamic potential in case of spontaneous symmetry breaking, recovered in other treatments through nonperturbative configurations like instantons.

To study the problem at hand, we combine the ideas of the perturbative functional RG for disordered systems with the general formalism of the exact/functional/nonperturbative RG. In the following, we shall denote our approach *nonperturbative functional RG* (NP-FRG). It provides a framework to study both perturbative and nonperturbative effects in any spatial dimension d and for any number of components of the fundamental fields, N . We exclude from the scope of the present series of articles relaxation and out-of-equilibrium dynamic phenomena, as well as spin glass

ordering. We also postpone to a forthcoming publication the development of the NP-FRG in a superfield formalism able to directly address the failure of supersymmetry in connection with that of dimensional reduction. Short versions of the present work have appeared in Refs. [30,39].

The present paper is organized as follows.

In section II we present the models and the formalism. We first introduce the models and discuss their physical relevance and the main open questions. From the corresponding replica field theories, we then derive the exact RG equation for the effective average action, which is the generating functional of the one-particle irreducible correlation functions at the running scale. We next relate the replica formalism, in which the replica symmetry is explicitly broken through the application of sources, to the cumulants of the renormalized disorder. We close the section by writing down the exact RG flow equations for these cumulants.

In section III, we introduce a systematic nonperturbative approximation scheme. After first discussing the symmetries of the problem and the way to implement them in the effective average action formalism, we introduce the nonperturbative truncation scheme of the exact RG equation: it relies on (i) an expansion in cumulants of the disorder and (ii) a well tested approximation of the nonperturbative RG, the "derivative expansion", which uses the fact that the relevant physics is dominated by long wavelength modes to perform an expansion in the number of spatial derivatives of the fundamental fields. Finally, we detail the minimal truncation that we use in our numerical investigation of the random field $O(N)$ model (RFO(N)M).

In section IV, we specialize the formalism to the study of the RFO(N)M. We introduce the scaling dimensions suitable to a search for the putative zero-temperature fixed point controlling the ordering transition. We first consider the case of the RFIM and then extend our description to the RFO(N)M. With the help of these dimensions, the RG flow equations are then cast in a scaled form. We also briefly comment on possible application to other disordered systems.

We next discuss in section V an important property of the truncations previously described: because of the one-loop structure of the exact flow equations and of the appropriate choice of the approximations, one recovers the perturbative results both near the upper critical dimension, $d_{uc} = 6$, and in the $N \rightarrow \infty$ limit of the RFO(N)M. Even more interestingly, we also show that our minimal truncation near the lower critical dimension for ferromagnetism of the RFO($N > 1$)M, $d_{lc} = 4$, reduces to the perturbative functional RG result (at one loop) obtained from the nonlinear sigma version of the model.²³ To the least, the truncated NP-FRG thus provides a nonperturbative interpolation in the whole (N, d) plane of the known perturbative results near $d = 4$, $d = 6$, as well as $N \rightarrow \infty$.

Finally, the presentation and discussion of the results

obtained for the $RFO(N)M$ within the present NP-FRG approach will be described in the companion paper.³²

II. MODELS AND FORMALISM

A. Models

We focus on the equilibrium, long-distance behavior of a class of disordered models in which N -component classical variables with $O(N)$ symmetric interactions are coupled to a random field. Depending on whether the coupling is linear or bilinear, the models belong to the “random field” (RF) or the “random anisotropy” (RA) subclasses. Such models with $N = 1, 2$, or 3 are relevant to describe a variety of systems encountered in condensed matter physics or physical chemistry. To name a few, one can mention dilute antiferromagnets in a uniform magnetic field,⁴⁰ critical fluids and binary mixtures in aerogels (both systems being modelled by the $N = 1$ RF Ising model),^{41,42,43} vortex phases in disordered type-II superconductors (described in terms of an elastic glass model whose simplest version is the $N = 2$ RF XY model),^{44,45,46} amorphous magnets, such as alloys of rare-earth compounds,^{47,48} and nematic liquid crystals in disordered porous media (described by $N = 2$ or $N = 3$ RA models).⁴⁹

Other related models can be described as well within the same formalism, but will only be alluded to: the “random elastic” model describing an elastic system, such as an interface or a vortex lattice, pinned by the presence of impurities; the “random temperature” model associated with impurity-generated bond or site dilution in a ferromagnetic Ising model. For reasons that will become clear further down in this section, we exclude from the present study spin glass ordering and we rather concentrate on ferromagnetic ordering (in which the $O(N)$ symmetry is spontaneously broken) or “quasi-ordering” (phases with quasi-long range order).

Our starting point is the field theoretical (coarse-grained) description of the systems in terms of an N -component scalar field $\chi(\mathbf{x})$ in a d -dimensional space and an effective Hamiltonian, or bare action,

$$S[\chi; \mathbf{h}, \boldsymbol{\tau}] = \int_{\mathbf{x}} \left\{ \frac{1}{2} \sum_{\mu=1}^N (|\partial \chi^\mu(\mathbf{x})|^2 + \tau \chi^\mu(\mathbf{x})^2) + \frac{u}{4!} \left(\sum_{\mu=1}^N \chi^\mu(\mathbf{x})^2 \right)^2 - \sum_{\mu=1}^N h^\mu(\mathbf{x}) \chi^\mu(\mathbf{x}) - \sum_{\mu, \nu=1}^N \tau^{\mu\nu}(\mathbf{x}) \chi^\mu(\mathbf{x}) \chi^\nu(\mathbf{x}) \right\}, \quad (1)$$

where $\int_{\mathbf{x}} \equiv \int d^d x$ and the superscript μ spans the N components of the field; $\mathbf{h}(\mathbf{x})$ is a random magnetic field and $\boldsymbol{\tau}(\mathbf{x})$ a second-rank random anisotropy tensor, which

are both taken for simplicity (see also discussion below) with gaussian distributions characterized by zero means and variances given by

$$\overline{h^\mu(\mathbf{x}) h^\nu(\mathbf{y})} = \Delta \delta_{\mu\nu} \delta(\mathbf{x} - \mathbf{y}) \quad (2)$$

$$\overline{\tau^{\mu\nu}(\mathbf{x}) \tau^{\rho\sigma}(\mathbf{y})} = \frac{\Delta_2}{2} (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) \delta(\mathbf{x} - \mathbf{y}), \quad (3)$$

where the overbar generically denotes the average over quenched disorder. Higher-order random anisotropies could be included as well. They will indeed be generated along the RG flow. However, for symmetry reasons, when starting with only a second-rank, or more generally an even-rank, random anisotropy, only even-rank anisotropies are generated: this corresponds to what is called the random anisotropy (RA) model. The model with a nonzero Δ , for which anisotropies of both odd and even ranks are generated under RG flow, is the random field (RF) model.

The equilibrium properties of the model are obtained from the average over disorder of the logarithm of the partition function,

$$\mathcal{Z}[\mathbf{J}] = \int \mathcal{D}\chi \exp \left(-S[\chi; \mathbf{h}, \boldsymbol{\tau}] + \int_{\mathbf{x}} \mathbf{J}(\mathbf{x}) \cdot \chi(\mathbf{x}) \right), \quad (4)$$

where $\mathbf{J}(\mathbf{x})$ is a source linearly coupled to the fundamental field and a (ultra-violet) momentum cutoff Λ , associated with an inverse microscopic lengthscale such as a lattice spacing, is implicitly considered in the functional integration over the field. With this definition however, the partition function and the corresponding thermodynamic potential $W[\mathbf{J}] = \ln \mathcal{Z}[\mathbf{J}]$ are still functionals of the random fields: $W[\mathbf{J}] \equiv W[\mathbf{J}; \mathbf{h}, \boldsymbol{\tau}]$. As is well known from the theory of systems with quenched disorder, the thermodynamics is given by the average over disorder of the “free energy”, *i.e.*,

$$\overline{W[\mathbf{J}]} = \overline{\ln \mathcal{Z}[\mathbf{J}]}. \quad (5)$$

Full information on the system, in particular an access to the correlation (Green) functions of the field, requires knowledge of the higher moments of $W[\mathbf{J}]$, viewed as a random functional.⁷⁴ As will be discussed more thoroughly further below, such information can be conveniently extracted by using the replica formalism whose starting point is the replacement of $\ln \mathcal{Z}$ by the limit of $(\mathcal{Z}^n - 1)/n$ when n , the number of replicas of the original system, goes to zero. Quite differently from the standard but controversial use of this replica trick, in which the analytic continuation for $n < 1$ opens the possibility of a *spontaneous* breaking of the replica symmetry,¹⁶ we will consider an *a priori* more benign procedure in which the symmetry between replicas is *explicitly* broken by the introduction of external sources acting on each replica independently. This procedure will allow us to generate the

cumulant expansion of the disorder-dependent functional $W[\mathbf{J}]$.

Within the replica formalism, the original problem is replaced by one with n replica fields $\{\chi_a(\mathbf{x})\}$, $a = 1, 2, \dots, n$, and the “replicated action”, obtained after explicitly performing the average over the disorder in the partition function, reads:

$$S_n[\{\chi_a\}] = \int_{\mathbf{x}} \left\{ \frac{1}{2} \sum_{a=1}^n [|\partial \chi_a(\mathbf{x})|^2 + \tau |\chi_a(\mathbf{x})|^2 + \frac{u}{12} (|\chi_a(\mathbf{x})|^2)^2] - \frac{1}{2} \sum_{a,b=1}^n [\Delta \chi_a(\mathbf{x}) \cdot \chi_b(\mathbf{x}) + \Delta_2(\chi_a(\mathbf{x}) \cdot \chi_b(\mathbf{x}))^2] \right\} \quad (6)$$

with the corresponding partition function

$$\mathcal{Z}_n[\{\mathbf{J}_a\}] = \int \prod_{a=1}^n \mathcal{D}\chi_a \exp \left(-S_n[\{\chi_a\}] + \sum_{a=1}^n \int_{\mathbf{x}} \mathbf{J}_a(\mathbf{x}) \cdot \chi_a(\mathbf{x}) \right) \quad (7)$$

where the linear sources $\mathbf{J}_a(\mathbf{x})$, $a = 1, 2, \dots, n$, act on each replica separately. Associated to this partition function is the generating functional of the connected Green functions, $W_n[\{\mathbf{J}_a\}] = \ln \mathcal{Z}_n[\{\mathbf{J}_a\}]$, and the effective action, $\Gamma_n[\{\phi_a\}]$, defined through a Legendre transform:

$$\Gamma_n[\{\phi_a\}] = -W_n[\{\mathbf{J}_a\}] + \sum_{a=1}^n \int_{\mathbf{x}} \mathbf{J}_a(\mathbf{x}) \cdot \phi_a(\mathbf{x}), \quad (8)$$

the fields $\{\phi_a\}$ and the sources $\{\mathbf{J}_a\}$ being related by

$$\phi_a^\mu(\mathbf{x}) = \langle \chi_a^\mu(\mathbf{x}) \rangle = \frac{\delta W_n[\{\mathbf{J}_a\}]}{\delta J_a^\mu(\mathbf{x})} \quad (9a)$$

where $\langle X \rangle$ represents the average of X with the weight given in Eq. (7), and

$$J_a^\mu(\mathbf{x}) = \frac{\delta \Gamma_n[\{\phi_a\}]}{\delta \phi_a^\mu(\mathbf{x})}. \quad (9b)$$

The effective action is the generating functional of the one-particle irreducible ($1 - PI$) correlation functions or proper vertices.

The formalism we are about to describe also applies to extensions of the replicated action of Eq. (6) that can be cast in the form

$$S_n[\{\chi_a\}] = \int_{\mathbf{x}} \left\{ \sum_{a=1}^n \left[\frac{1}{2} |\partial \chi_a(\mathbf{x})|^2 + U_\Lambda(\chi_a(\mathbf{x})) \right] - \frac{1}{2} \sum_{a,b=1}^n V_\Lambda(\chi_a(\mathbf{x}), \chi_b(\mathbf{x})) + \dots \right\}, \quad (10)$$

where the subscript Λ recalls that the various terms are at their bare value, defined at the microscopic scale Λ , and the dots indicate possible functions involving higher numbers of replicas. The functions $U_\Lambda, V_\Lambda, \dots$ satisfy the $O(N)$ symmetry as well as the S_n permutational symmetry between replicas. Eq. (1) is obviously a special case of the above expression, and higher-order anisotropies are included in a 2-replica term which is only function of $\chi_a(\mathbf{x}) \cdot \chi_b(\mathbf{x})$. RF and RA $O(N)$ models with nongaussian distributions of the random fields and anisotropies are described by terms involving higher number of replicas. (Note that the RA $O(N)$ model is defined as such for $N > 1$; the Ising case, $N = 1$, corresponds to another model, the random temperature one introduced hereafter.)

Other disordered systems are also described by the form of the replicated action in Eq. (10). For instance, the random temperature model corresponds to Eq. (10) with U_Λ and V_Λ functions of the fields only through the $O(N)$ invariants $\rho_a = \frac{1}{2} |\chi_a|^2, \rho_b = \frac{1}{2} |\chi_b|^2$. In the RF, RA, and random temperature models, the 1-replica part of the bare action simply describes n copies of the standard ferromagnetic $O(N)$ model without disorder.

The random elastic model is also a special case of Eq. (10). However, contrary to the models just discussed, the 1-replica potential U_Λ is absent (or reduced to a purely quadratic term), so that there is no mechanism triggering a paramagnetic-ferromagnetic phase transition. The 2-replica potential V_Λ , which is the second cumulant of a random pinning potential, is now function of only the difference between the two replica fields, $\chi_a(\mathbf{x}) - \chi_b(\mathbf{x})$. As a result, the model has an additional symmetry, the statistical tilt symmetry,⁵⁰ which guarantees that the 1-replica part of the action, including the kinetic term, is not renormalized: the effective action has thus the same 1-replica part as the bare one. (Note that, as shown in Ref. [30] and in the companion paper,³² the random elastic model, albeit with an underlying periodicity, also emerges as a low-disorder approximation of the RF and RA XY ($N = 2$) models.)

B. Exact RG equation for the effective average action

The exact RG in the effective average action formalism^{34,36,51} relates the bare action, here Eq. (10), to the full effective action, Eq. (8), through a progressive inclusion of fluctuations of longer and longer wavelength. To do so, one introduces an infrared regulator, characterized by a scale k , which, in the functional integration leading to the partition function, suppresses the contribution of the low-energy modes with momentum $|\mathbf{q}| \lesssim k$ while including the high-energy modes with $|\mathbf{q}| \gtrsim k$. After Legendre transformation, this defines an “effective average action” at the running scale k , Γ_k , which continuously interpolates between the microscopic scale $k = \Lambda$, at which $\Gamma_{k=\Lambda}$ reduces to the bare action, and the macro-

scopic one, $k = 0$, at which $\Gamma_{k=0}$ equals the full effective action.

More precisely in the present context, a “mass-like” quadratic term is added to the bare action, Eq. (10),

$$\Delta\mathcal{S}_k[\{\chi_a\}] = \frac{1}{2} \sum_{a,b=1}^n \sum_{\mu,\nu=1}^N \int_{\mathbf{q}} R_{k,ab}^{\mu\nu}(q^2) \chi_a^\mu(-\mathbf{q}) \chi_b^\nu(\mathbf{q}), \quad (11)$$

where $\int_{\mathbf{q}} \equiv \int d^d q / (2\pi)^d$; $R_{k,ab}^{\mu\nu}(q^2)$ denotes infrared cut-off functions which, in order to enforce that the additional term satisfies the same $O(N)$ and S_n symmetries as the bare action (see above), must take the following form:

$$R_{k,ab}^{\mu\nu}(q^2) = \left(\hat{R}_k(q^2) \delta_{ab} + \tilde{R}_k(q^2) \right) \delta_{\mu\nu}. \quad (12)$$

The cutoff functions $\hat{R}_k(q^2)$ and $\tilde{R}_k(q^2)$ are chosen such as to realize the decoupling of the low- and high-momentum modes at the scale k : for this, they must decrease sufficiently fast for large momentum $|\mathbf{q}| \gg k$ and go to a constant value (a “mass”) for small momentum $|\mathbf{q}| \ll k$. The presence of an off-diagonal component $\tilde{R}_k(q^2)$ is somewhat unusual and will be discussed later on. The cutoff functions must also satisfy the two constraints that (i) they go to zero when $k \rightarrow 0$, so that one indeed recovers the full effective action with all modes accounted for, and (ii) $\hat{R}_k(q^2)$ diverges while $\tilde{R}_k(q^2)$ stays finite when $k \rightarrow \Lambda$, so that the effective average action does reduce to the bare action. (In what follows we are only concerned with the long-distance behavior of the models and do not pay attention to microscopic details; we thus let Λ go to ∞ in the cutoff functions.) Different choices have been proposed and tested in the recent literature. Standard choices for $\hat{R}_k(q^2)$ are of the form

$$\hat{R}_k(q^2) = Z_k q^2 r(q^2/k^2) \quad (13)$$

with Z_k a field renormalization constant yet to be specified and $r(y) = y^{-1}(1-y)\Theta(1-y)$,⁵² where Θ is the Heaviside function, or $r(y) = (e^y - 1)^{-1}$.⁵¹

From the partition function $\mathcal{Z}_k[\{\mathbf{J}_a\}]$ obtained from the bare action supplemented with the k -dependent regulator, Eq. (11), one defines the generating functional of the Green functions $W_k[\{\mathbf{J}_a\}] = \ln \mathcal{Z}_k[\{\mathbf{J}_a\}]$ and, through a Legendre transform, one has access to the effective average action at the running scale k , Γ_k :

$$\Gamma_k[\{\phi_a\}] + W_k[\{\mathbf{J}_a\}] = \sum_{a=1}^n \int_{\mathbf{x}} \mathbf{J}_a(\mathbf{x}) \cdot \phi_a(\mathbf{x}) - \Delta\mathcal{S}_k[\{\phi_a\}] \quad (14)$$

where the fields $\{\phi_a\}$ and the sources $\{\mathbf{J}_a\}$ are related by the (k -dependent) expression

$$\phi_a^\mu(\mathbf{x}) = \langle \chi_a^\mu(\mathbf{x}) \rangle = \frac{\delta W_k[\{\mathbf{J}_a\}]}{\delta J_a^\mu(\mathbf{x})}. \quad (15)$$

The Legendre transform is slightly modified by the addition of the last in Eq. (14), which ensures that the

effective average action Γ_k does reduce to the bare action at the microscopic scale, with no contribution from the infrared regulator. This addition does not change the behavior in the $k \rightarrow 0$ limit since the regulator goes identically to zero. Physically, and to use the language of magnetic systems, the effective average action is a coarse-grained Gibbs free energy. It is the generating functional of the $1 - PI$ correlation functions from which one can derive all Green functions of the modified system at the scale k . Note that here and in the following we omit the subscript n associated to the number of replicas in order to simplify the notations.

The evolution of the effective average action with the infrared cutoff k is governed by an exact flow equation,

$$\partial_k \Gamma_k[\{\phi_a\}] = \frac{1}{2} \int_{\mathbf{q}} Tr \left\{ \partial_k \mathbf{R}_k(q^2) \left[\Gamma_k^{(2)} + \mathbf{R}_k \right]_{\mathbf{q}, -\mathbf{q}}^{-1} \right\}, \quad (16)$$

where the trace involves a sum over both replica indices and N -vector components; $\mathbf{R}_k(q^2)$ is defined in Eq. (12) and $\Gamma_k^{(2)}$ is the tensor formed by the second functional derivatives of Γ_k with respect to the fields $\phi_a^\mu(\mathbf{q})$:

$$\left(\Gamma_k^{(2)} \right)_{ab}^{\mu\nu}(\mathbf{q}, \mathbf{q}') = \frac{\delta^2 \Gamma_k}{\delta \phi_a^\mu(\mathbf{q}) \delta \phi_b^\nu(\mathbf{q}')}. \quad (17)$$

The above RG flow equation is a complicated functional integro-differential equation that cannot be solved exactly in general; but, due to its one-loop structure and its reasonably transparent physical content, it provides a convenient starting point for nonperturbative approximation schemes.

At this point, it is quite clear to see why we have excluded spin glass ordering from our considerations. The quadratic form of the infrared regulator in Eq. (11) suppresses the fluctuations of the low-momentum modes of the fundamental fields χ_a . Spin glass ordering on the other hand involves fluctuations of composite fields, associated *e.g.* to the “overlap” between different replicas.¹⁶ Proper RG treatment of such fluctuations implies to introduce a “mass-like” regulator for composite fields, *i.e.*, in the simplest case a functional that is quartic in the fundamental fields instead of the quadratic term used here. We do not consider this case in the present work.

C. Explicit replica symmetry breaking and cumulants of the renormalized disorder

Among the technical difficulties encountered when making use of the exact RG equation, Eq. (16), there is one which is specific to disordered systems and to the present replica formalism: one must invert the matrix $\Gamma_{k,ab}^{(2)} + \mathbf{R}_{k,ab}$ for arbitrary replica fields (since all replicas are different due to the independently applied sources). Before delving into this problem, it is worth giving some physical insight into the meaning of the explicit replica symmetry breaking used here.

As discussed in section II-A, after full account of the fluctuations, the bare disorder is renormalized to a full random (“free energy”) functional $W[\mathbf{J}]$, which, to make its dependence on the bare quenched disorder explicit, we now denote $W[\mathbf{J}; \mathbf{h}]$. This random object can be characterized by the infinite set of its cumulants, $W_1[\mathbf{J}_1], W_2[\mathbf{J}_1, \mathbf{J}_2], W_3[\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3], \dots$, with

$$W_1[\mathbf{J}_1] = \overline{W[\mathbf{J}_1; \mathbf{h}]} \quad (18)$$

$$W_2[\mathbf{J}_1, \mathbf{J}_2] = \overline{W[\mathbf{J}_1; \mathbf{h}]W[\mathbf{J}_2; \mathbf{h}]} - \overline{W[\mathbf{J}_1; \mathbf{h}]} \overline{W[\mathbf{J}_2; \mathbf{h}]}, \quad (19)$$

etc... The first cumulant W_1 gives access to the thermodynamics of the system and the higher-order cumulants describe the distribution of the renormalized disorder (we define, as in the bare action, a disorder with zero mean). Note that by construction the cumulants are invariant under permutations of their arguments.

The cumulants can be generated from an average involving copies, or “replicas”, of the original disordered system, as follows:

$$\begin{aligned} \overline{\exp\left(\sum_{a=1}^n W[\mathbf{J}_a; \mathbf{h}]\right)} &= \exp(W[\{\mathbf{J}_a\}]) \\ &= \exp\left(\sum_{a=1}^n W_1[\mathbf{J}_a] + \frac{1}{2} \sum_{a,b=1}^n W_2[\mathbf{J}_a, \mathbf{J}_b] \right. \\ &\quad \left. + \frac{1}{3!} \sum_{a,b,c=1}^n W_3[\mathbf{J}_a, \mathbf{J}_b, \mathbf{J}_c] + \dots\right), \end{aligned} \quad (20)$$

where the n copies have the *same* bare disorder but are coupled to different external sources. To fully characterize the random functional $W[\mathbf{J}; \mathbf{h}]$, it is indeed important to describe its cumulants for generic arguments, *i.e.*, for different sources. (Be aware that the subscripts 1, 2, ... used to denote the cumulants of W should not be confused with the subscript n denoting the number of replicas in section II-A and omitted since: here for instance, W_1 denotes the 1-replica component, corresponding to the first cumulant, whereas with the previous notation $W_{n=1}$ is given by the sum of all cumulants with all there arguments equal.)

A convenient trick to extract the cumulants with their full functional dependence is to let the number of replicas be arbitrary and to view the expansion in the right-hand side of Eq. (20) as an expansion in increasing number of “free”, or unconstrained, sums over replicas of the functional $W[\{\mathbf{J}_a\}]$ defined below Eq. (7). The term of order p in the expansion is a sum over p replica indices of a functional depending exactly on p replica sources, this functional being precisely equal here to the p th cumulant of $W[\mathbf{J}; \mathbf{h}]$. This procedure, which rests on an explicit breaking of the replica symmetry and an analytic continuation to arbitrary numbers of replicas (including the limit $n \rightarrow 0$ previously introduced), is *a priori* different from the standard use of replicas, in which all sources

are equal, and it avoids the delicate handling of a spontaneous replica symmetry breaking.^{1,2,16,17} It has been used in a similar context by Le Doussal and Wiese.^{53,54} The practical implementation of the expansion in free replica sums will be detailed in the next subsection.

In our present NP-FRG approach however, the central object is the effective action Γ , not W . The expansion of $\Gamma[\{\phi_a\}]$ in increasing number of free replica sums reads

$$\begin{aligned} \Gamma[\{\phi_a\}] &= \sum_{a=1}^n \Gamma_1[\phi_a] - \frac{1}{2} \sum_{a,b=1}^n \Gamma_2[\phi_a, \phi_b] \\ &\quad + \frac{1}{3!} \sum_{a,b,c=1}^n \Gamma_3[\phi_a, \phi_b, \phi_c] + \dots, \end{aligned} \quad (21)$$

where for later convenience we have introduced a minus sign for all even terms of the expansion. $\Gamma[\{\phi_a\}]$ and $W[\{\mathbf{J}_a\}]$ are related by a Legendre transform, so if one also expand the sources $\mathbf{J}_a[\{\phi_f\}]$ (where we have denoted $\{\phi_f\}$ the n replica fields to avoid confusion in the indices) in increasing number of free replica sums, one can relate the terms of the expansion of the effective action to the cumulants of the random functional $W[\mathbf{J}; \mathbf{h}]$. The relation is straightforward for the first terms, but gets more involved as the order increases.

More precisely, $\Gamma_1[\phi]$ is the Legendre transform of $W_1[\mathbf{J}]$, namely,

$$\Gamma_1[\phi] = -W_1[\mathbf{J}] + \int_{\mathbf{x}} \mathbf{J}(\mathbf{x}) \cdot \phi(\mathbf{x}), \quad (22)$$

with

$$\phi^\mu(\mathbf{x}) = \frac{\delta W_1[\mathbf{J}]}{\delta J^\mu(\mathbf{x})}, \quad (23)$$

and the second-order terms is given by

$$\Gamma_2[\phi_1, \phi_2] = W_2[\mathbf{J}[\phi_1], \mathbf{J}[\phi_2]], \quad (24)$$

where $\mathbf{J}[\phi]$ is the *nonrandom* source defined via the inverse of the Legendre transform relation in Eq. (22), *i.e.*, $J^\mu[\phi](\mathbf{x}) = \delta \Gamma_1[\phi] / \delta \phi^\mu(\mathbf{x})$. (Note that $\mathbf{J}(\mathbf{x})$ introduced here differs from the source $\mathbf{J}_a(\mathbf{x})$ introduced in equation (9b): through the Legendre relations, the latter depends on all the fields $\{\phi_a\}$ while the former depends on only one replica field.) The above expression motivates our choice of signs for the terms of the expansion in free replica sums of $\Gamma[\{\phi_a\}]$, Eq. (20): $\Gamma_2[\phi_1, \phi_2]$ is directly the second cumulant of $W[\mathbf{J}; \mathbf{h}]$ (with the proper choice of $\mathbf{J}[\phi]$).

For the higher-order terms, one finds

$$\begin{aligned} \Gamma_3[\phi_1, \phi_2, \phi_3] &= -W_3[\mathbf{J}[\phi_1], \mathbf{J}[\phi_2], \mathbf{J}[\phi_3]] + \\ &\quad \int_{\mathbf{x}\mathbf{y}} \left\{ W_{2,\mathbf{x}}^{(10)}[\mathbf{J}[\phi_1], \mathbf{J}[\phi_2]] \left(W_1^{(2)}[\mathbf{J}[\phi_1]] \right)^{-1}_{\mathbf{x}\mathbf{y}} \right. \\ &\quad \left. \times W_{2,\mathbf{y}}^{(10)}[\mathbf{J}[\phi_1], \mathbf{J}[\phi_3]] + \text{perm}(123) \right\}, \end{aligned} \quad (25)$$

etc..., where $\text{perm}(123)$ denotes the two additional terms obtained by circular permutations of the fields ϕ_1, ϕ_2, ϕ_3 and where we have used the following short-hand notation:

$$W_{1, \mathbf{x}_1 \dots \mathbf{x}_p}^{(p)}[\mathbf{J}_1] = \frac{\delta^p W_1[\mathbf{J}_1]}{\delta J_1(\mathbf{x}_1) \dots \delta J_1(\mathbf{x}_p)}, \quad (26)$$

$$W_{2, \mathbf{x}_1 \dots \mathbf{x}_p, \mathbf{y}_1 \dots \mathbf{y}_q}^{(pq)}[\mathbf{J}_1, \mathbf{J}_2] = \frac{\delta^{p+q} W_2[\mathbf{J}_1, \mathbf{J}_2]}{\delta J_1(\mathbf{x}_1) \dots \delta J_1(\mathbf{x}_p) \delta J_2(\mathbf{y}_1) \dots \delta J_2(\mathbf{y}_q)}, \quad (27)$$

etc. Note that for clarity the $O(N)$ indices have been omitted in the above expressions.

We point out that $\Gamma_p[\phi_1, \dots, \phi_p]$ for $p \geq 3$ cannot be directly taken as the p th cumulant of a physically accessible random functional, in particular not of the disorder-dependent Legendre transform of $W[\mathbf{J}; \mathbf{h}]$ (although it can certainly be expressed in terms of such cumulants of order equal or lower than p). In the following and by abuse of language, we will nonetheless generically call the Γ_p 's "cumulants of the renormalized disorder" (which is true for $p = 2$).

In complement to the above picture and more specifically for random field systems, it is also interesting to introduce a renormalized random field (or random force) $\check{h}[\phi](\mathbf{x})$ defined as the derivative of a random free-energy functional,

$$\check{h}[\phi]^\mu(\mathbf{x}) = -\frac{\delta}{\delta \phi^\mu(\mathbf{x})} \left(W[\mathbf{J}[\phi]; \mathbf{h}] - \overline{W[\mathbf{J}[\phi]; \mathbf{h}]} \right), \quad (28)$$

and whose first moment is equal to zero by construction. It is easy to derive that its p th cumulant ($p \geq 2$) is given by the derivative with respect to ϕ_1, \dots, ϕ_p of $W_p[\mathbf{J}[\phi_1], \dots, \mathbf{J}[\phi_p]]$, which can then be related to derivatives of $\Gamma_2, \Gamma_3, \dots$; for instance,

$$\overline{\check{h}[\phi_1](\mathbf{x}) \check{h}[\phi_2](\mathbf{y})} = \Gamma_{2, \mathbf{x}\mathbf{y}}^{(11)}[\phi_1, \phi_2], \quad (29)$$

where we have used a short-hand notation similar to that of Eqs. (26,27) and omitted the N -vector indices for simplicity. Terms of order 3 and higher are again given by more complicated expressions.

We close this discussion by noticing that in the simpler case of the random manifold model, Γ_1 and W_1 being trivial and unrenormalized due to the statistical tilt symmetry (see above), $\mathbf{J}[\phi]$ has a simple explicit expression. For instance, if the bare action has a quadratic 1-replica term, $\Gamma_1[\phi]$ is equal to this quadratic functional and $\mathbf{J}[\phi]$ is a known linear functional of ϕ , which further simplifies when considering uniform fields. This allows one to devise ways to directly measure the second cumulant of the renormalized disorder.^{55,56} Nothing similar occurs in random field and random anisotropy models: the thermodynamics of such systems being highly nontrivial (with a phase transition and a critical point), the expression of $\mathbf{J}[\phi]$ is involved and a priori unknown.

D. Exact RG equations for the renormalized disorder cumulants

The reasoning developed in the previous subsection can be applied to the effective average action Γ_k and its expansion in free replica sums. As a result, Eqs. (18-29) can be extended to any running scale k . Yet, to make the expansion in free replica sums an operational procedure, one needs to be able to perform systematic algebraic manipulations, as for instance the inversion of the matrix appearing in the right-hand side of the exact RG equation, Eq. (16). We detail here the method for matrices depending on two replica indices, but functionals of the n replica fields. Extension to higher-order tensors is presented in Ref. [54].

A generic matrix $A_{ab}[\{\phi_f\}]$, where we have again denoted $\{\phi_f\}$ the n replica fields to avoid confusion in the indices, can be decomposed as

$$A_{ab}[\{\phi_f\}] = \hat{A}_a[\{\phi_f\}] \delta_{ab} + \tilde{A}_{ab}[\{\phi_f\}]. \quad (30)$$

In the above expression, it is understood that the second term \tilde{A}_{ab} no longer contains any Kronecker symbol. Each component can now be expanded in increasing number of free replica sums,

$$\hat{A}_a[\{\phi_f\}] = \hat{A}^{[0]}[\phi_a] + \sum_{c=1}^n \hat{A}^{[1]}[\phi_a | \phi_c] + \dots \quad (31)$$

$$\tilde{A}_{ab}[\{\phi_f\}] = \tilde{A}^{[0]}[\phi_a, \phi_b] + \sum_{c=1}^n \tilde{A}^{[1]}[\phi_a, \phi_b | \phi_c] + \dots, \quad (32)$$

where the superscripts in square brackets denote the order in the expansion (and should not be confused with superscripts in parentheses indicating partial derivatives).

As an illustration, the expansion of the matrix $\mathbf{\Gamma}_k^{(2)}$ defined in Eq. (17) reads, in terms of the expansion of effective average action itself,

$$\hat{\mathbf{\Gamma}}_k^{(2)}[\{\phi_f\}]_a = \mathbf{\Gamma}_{k,1}^{(2)}[\phi_a] - \sum_{c=1}^n \mathbf{\Gamma}_{k,2}^{(20)}[\phi_a, \phi_c] + \dots \quad (33)$$

$$\tilde{\mathbf{\Gamma}}_k^{(2)}[\{\phi_f\}]_{ab} = -\mathbf{\Gamma}_{k,2}^{(11)}[\phi_a, \phi_b] + \sum_{c=1}^n \mathbf{\Gamma}_{k,3}^{(110)}[\phi_a, \phi_b, \phi_c] + \dots, \quad (34)$$

where the permutational symmetry of the arguments of the $\mathbf{\Gamma}_{k,p}$'s has been used.

Algebraic manipulations on such matrices can be performed by term-by-term identification of the orders of the expansions. For instance, the inverse $\mathbf{B} = \mathbf{A}^{-1}$ of the matrix \mathbf{A} can also be put in the form of Eq. (30) and its components, \hat{B}_a and \tilde{B}_{ab} , expanded in number of free replica sums. The term-by-term identification of the

condition $\mathbf{A} \cdot \mathbf{B} = \mathbf{1}$ leads to a unique expression of the various orders, $\widehat{B}^{[p]}$ and $\widetilde{B}^{[p]}$, of the expansion of \mathbf{B} in terms of the $\widehat{A}^{[q]}$'s and $\widetilde{A}^{[q]}$'s with $q \leq p$. The algebra becomes rapidly tedious, but the first few terms are easily derived:

$$\widehat{B}^{[0]}[\phi_1] = \widehat{A}^{[0]}[\phi_1]^{-1} \quad (35)$$

$$\widetilde{B}^{[0]}[\phi_1, \phi_2] = -\widehat{B}^{[0]}[\phi_1] \widetilde{A}^{[0]}[\phi_1, \phi_2] \widehat{B}^{[0]}[\phi_2] \quad (36)$$

$$\widehat{B}^{[1]}[\phi_1 | \phi_2] = -\widehat{B}^{[0]}[\phi_1] \widehat{A}^{[1]}[\phi_1 | \phi_2] \widehat{B}^{[0]}[\phi_1] \quad (37)$$

$$\begin{aligned} \widetilde{B}^{[1]}[\phi_1, \phi_2 | \phi_3] = & -\widehat{B}^{[0]}[\phi_1] \left\{ \widetilde{A}^{[1]}[\phi_1, \phi_2 | \phi_3] \right. \\ & - \widetilde{A}^{[0]}[\phi_1, \phi_3] \widehat{B}^{[0]}[\phi_3] \widetilde{A}^{[0]}[\phi_3, \phi_2] \\ & - \widehat{A}^{[1]}[\phi_1 | \phi_3] \widehat{B}^{[0]}[\phi_1] \widetilde{A}^{[0]}[\phi_1, \phi_2] \\ & \left. - \widetilde{A}^{[0]}[\phi_1, \phi_2] \widehat{B}^{[0]}[\phi_2] \widehat{A}^{[1]}[\phi_2 | \phi_3] \right\} \widehat{B}^{[0]}[\phi_2] \end{aligned} \quad (38)$$

etc.

We can apply the above procedure to the exact RG equation for the effective average action. For convenience, we introduce the modified propagator at the scale k ,

$$\mathbf{P}_k[\{\phi_f\}] = \left[\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k \right]^{-1}, \quad (39)$$

with

$$\mathbf{P}_{k,ab}[\{\phi_f\}] = \widehat{\mathbf{P}}_{k,a}[\{\phi_f\}] \delta_{ab} + \widetilde{\mathbf{P}}_{k,ab}[\{\phi_f\}], \quad (40)$$

where $\widehat{\mathbf{P}}_{k,a}$ and $\widetilde{\mathbf{P}}_{k,ab}$ are still tensors with respect to momenta and vector component indices. Eq. (16) then leads to an infinite hierarchy of flow equations for the cumulants of the renormalized disorder,

$$\begin{aligned} \partial_k \Gamma_{k,1}[\phi_1] = & \frac{1}{2} \int_q \left\{ \partial_k (\widehat{R}_k(q^2) + \widetilde{R}_k(q^2)) \text{tr} \widehat{\mathbf{P}}_{k,q-q}^{[0]}[\phi_1] \right. \\ & \left. + \partial_k \widehat{R}_k(q^2) \text{tr} \widetilde{\mathbf{P}}_{k,q-q}^{[0]}[\phi_1, \phi_1] \right\}, \end{aligned} \quad (41)$$

$$\begin{aligned} \partial_k \Gamma_{k,2}[\phi_1, \phi_2] = & -\frac{1}{2} \int_q \left\{ \partial_k (\widehat{R}_k(q^2) + \widetilde{R}_k(q^2)) \right. \\ & \text{tr} \widehat{\mathbf{P}}_{k,q-q}^{[1]}[\phi_1 | \phi_2] + \partial_k \widehat{R}_k(q^2) \text{tr} \widetilde{\mathbf{P}}_{k,q-q}^{[1]}[\phi_1, \phi_1 | \phi_2] \\ & \left. + \partial_k \widetilde{R}_k(q^2) \text{tr} \widetilde{\mathbf{P}}_{k,q-q}^{[0]}[\phi_1, \phi_2] + \text{perm}(12) \right\}, \end{aligned} \quad (42)$$

and so on, where tr indicates a trace over N -vector components and $\text{perm}(12)$ denotes the expression obtained by permuting ϕ_1 and ϕ_2 . (Some care is needed in the

term by term identification in order to properly symmetrize the expressions and satisfy the permutational property of the various arguments of the cumulants.)

Expressing the higher-order terms $\widehat{\mathbf{P}}_k^{(p)}$ and $\widetilde{\mathbf{P}}_k^{(p)}$ with $p \geq 1$ only by means of $\widehat{\mathbf{P}}_k^{[0]}$ and the derivatives of the $\Gamma_{k,p}$'s and introducing the short-hand notation $\widetilde{\partial}_k$ to indicate a derivative acting only on the cutoff functions, *i.e.*, $\widetilde{\partial}_k \equiv \partial_k \widehat{R}_k \delta / \delta \widehat{R}_k + \partial_k \widetilde{R}_k \delta / \delta \widetilde{R}_k$, Eq. (42) can be rewritten

$$\begin{aligned} \partial_k \Gamma_{k,2}[\phi_1, \phi_2] = & \frac{1}{2} \widetilde{\partial}_k \text{Tr} \left\{ \widehat{\mathbf{P}}_k^{[0]}[\phi_1] (\mathbf{\Gamma}_{k,2}^{(20)}[\phi_1, \phi_2] \right. \\ & - \mathbf{\Gamma}_{k,3}^{(110)}[\phi_1, \phi_1, \phi_2]) + \widetilde{\mathbf{P}}_k^{[0]}[\phi_1, \phi_1] \mathbf{\Gamma}_{k,2}^{(20)}[\phi_1, \phi_2] \\ & \left. + \frac{1}{2} \widetilde{\mathbf{P}}_k^{[0]}[\phi_1, \phi_2] (\mathbf{\Gamma}_{k,2}^{(11)}[\phi_1, \phi_2] - \widetilde{R}_k \mathbf{1}) + \text{perm}(12) \right\}, \end{aligned} \quad (43)$$

and similarly for higher-order cumulants, where $\mathbf{1}_{qq'}^{\mu\nu} = (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \delta_{\mu\nu}$ and the trace Tr is now over both momenta and N -vector components; the modified propagators $\widehat{\mathbf{P}}_k^{[0]}$ and $\widetilde{\mathbf{P}}_k^{[0]}$ are explicitly given by

$$\widehat{\mathbf{P}}_k^{[0]}[\phi_1] = \left(\mathbf{\Gamma}_{k,1}^{(2)}[\phi_1] + \widehat{R}_k \mathbf{1} \right)^{-1}, \quad (44)$$

$$\widetilde{\mathbf{P}}_k^{[0]}[\phi_1, \phi_2] = \widehat{\mathbf{P}}_k^{[0]}[\phi_1] (\mathbf{\Gamma}_{k,2}^{(11)}[\phi_1, \phi_2] - \widetilde{R}_k \mathbf{1}) \widehat{\mathbf{P}}_k^{[0]}[\phi_2] \quad (45)$$

This provides a hierarchy of exact RG equations for the cumulants of the renormalized disorder (including the first one which leads to a description of the thermodynamics). One should note that (i) the cumulants are functional of the fields and contain full information on the complete set of $1 - PI$ correlation functions and (ii) the flow equations are coupled, the $(p+1)$ th cumulant appearing in the right-hand side of the equation for the p th cumulant. As such these RG equations remain untractable and their resolution requires approximations.

III. NONPERTURBATIVE APPROXIMATION SCHEME

A. Symmetries in the effective average action formalism

When writing the RG flow for the effective average action and when devising an approximation scheme to solve it, one should as far as possible make sure that the symmetries of the theory are not explicitly violated at any scale. Such a requirement is easily implemented as far as elementary symmetries, such as invariance by translation and rotation in Euclidean space, $O(N)$ symmetry, and S_n replica permutational symmetry, are concerned: the infrared regulator $\Delta \mathcal{S}_k$ added to the bare action must be chosen such that it is invariant under the appropriate

transformations, which is indeed guaranteed by the expressions in Eqs. (11,12). The exact effective average action at any scale k then also possesses the symmetries of the bare action, and one just had to be careful that the truncations do not explicitly break the symmetries, which is easily implemented.³⁶

A similar treatment can be applied to most additional symmetries of the disordered systems under consideration. For instance, the “statistical tilt symmetry” of the random manifold model is easily extended to a k -dependent statistical tilt symmetry with any regulator of the form given in Eqs. (11,12), which implies that the 1-replica part (first cumulant) of the effective average action is unrenormalized along the flow. Similarly, the additional inversion symmetries of the random anisotropy ($\chi_a \cdot \chi_b \rightarrow -\chi_a \cdot \chi_b$) and the random temperature ($\chi_a, \{\chi_b\}_{b \neq a} \rightarrow -\chi_a, \{\chi_b\}_{b \neq a}$) models are readily accounted for with the choice $\bar{R}_k \equiv 0$. Truncation schemes naturally follow.

Taking into account the underlying supersymmetry that characterizes the random field model for a gaussian distribution of the random field⁷ is much more involved. First, because one knows that the supersymmetry, which goes with the dimensional reduction property, must be broken in low enough dimension (at least, in $d = 3$), so that, even if the RG flow is started with an initial condition obeying supersymmetry, a mechanism should be provided to describe a spontaneous breaking of the supersymmetry. Secondly, the supersymmetry shows up in a superfield formalism built with auxiliary fermionic and bosonic fields, but it is far from transparent in the present framework based on the fundamental fields. (This is true already at the level of the initial condition of the RG flow.) We shall therefore defer the proper resolution of this problem to a forthcoming publication.⁵⁷ Note that an underlying supersymmetry is also present in the random manifold model, where it also leads to the $d \rightarrow d - 2$ dimensional reduction. However, the pure model with no disorder is merely a free field theory, and this is easily accounted for.⁵⁸

B. Truncation schemes

We have already stressed that solving the exact RG equation for the effective average action requires approximations. The general framework has proven quite versatile for devising efficient and numerically tractable approximations which are able to describe both universal and nonuniversal properties in any spatial dimension and to capture genuine nonperturbative phenomena (see Introduction). Such approximations generally amount to truncating the functional form of the effective average action, which results in a self-consistent flow that preserves the fundamental structure of the theory (as the symmetries, see above).

If one is interested in the long-distance physics of a system and in observables at small momenta, a systematic

truncation scheme is provided by the so-called “*derivative expansion*”.^{34,36} It consists in expanding the effective average action in increasing number of derivatives of the field(s) and retaining only a limited number of terms. The lowest order is the “local potential approximation” (LPA)⁵⁹ in which one only considers the flow of the effective average potential, *i.e.*, the effective average action for a uniform field configuration. The field is not renormalized and the associated anomalous dimension is equal to zero. Field renormalization, which is important in the present problem where one expects the anomalous dimension to be quite sizeable in low dimensions (*e.g.*, numerical estimates give $\eta \simeq 0.5$ for the RFIM in $d = 3$), requires to go beyond the LPA and to consider the first order of the derivative expansion. Previous studies on a variety of systems, including the pure $O(N)$ model, have shown that the system’s behavior is quantitatively very well described at this level of approximation.^{36,37,60,61} Higher-order terms improve the accuracy,^{62,63} but they rapidly become untractable except in simple models.

For the disordered systems considered here, one more step is needed. We have seen in section II-C that an *expansion in number of free replica sums* can be used to generate the cumulants of the renormalized disorder. Keeping only a limited number of terms in the expansion therefore leads to a systematic truncation scheme. To describe both the thermodynamics and the renormalized probability distribution of the disorder, one must consider at least the first two cumulants, or equivalently, the second order in the expansion in free replica sums.

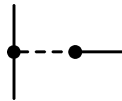
Finally, on top of the two previous approximations, it may be useful, and numerically more tractable, to expand the functions appearing in the truncated effective average action in powers of the field considered around a given (uniform) configuration. This configuration can be taken either as zero everywhere or as a nontrivial configuration that minimizes the effective average potential (here, more precisely, its 1-replica component that gives access to the thermodynamics). Again, the accuracy and convergence properties of such field expansions have been widely tested for many different models. In the present case, and for reasons that will become clear later on, field expansions should be used with great caution.

C. Minimal truncation

Given the general scheme presented above, the choice of a minimal nonperturbative truncation is guided by a combination of factors: experience gained from studies on other models, constraints associated with the symmetries of the full theory, intuition or previous knowledge concerning the physics of the problem at hand, requirement of being able to recover as much as possible exact and perturbative results in the appropriate limits, and of course, a practical limitation coming with the numerical capability to actually solve the set of RG flow equations.

As we have already alluded to, a description of the

where a line denotes the Goldstone propagator and dots represent vertices obtained from derivatives of either the 1-replica potential (single dots) or the 2-replica potential (dots linked by a dashed line); for instance,



represents the three-point vertex $\Gamma_k^{(21)} \equiv V_k^{(21)}$. We did not include the graphs containing 4-point vertices because in the truncation considered here, they do not contribute to the flow of $Z_{m,k}$. From the above flow equation, Eq. (50), one extracts a running anomalous exponent,

$$\eta_k = -k\partial_k Z_{m,k}. \quad (51)$$

The initial conditions for the RG flow equations are obtained from the bare action, Eq. (10). The RG flow equations form a closed set of coupled nonlinear integro-differential equations for two functions, $U_k(\rho_1)$ and $V_k(\rho_1, \rho_2, z)$, and a constant, $Z_{m,k}$. The numerical task of solving these equations is still arduous and when needed for reducing the difficulty of the computations, we will also consider truncated expansions of the 1- and 2-replica potentials in some or all of their field arguments (see below).

The present approach represents a nonperturbative but of course approximate RG description. Already at the minimal truncation discussed above, one includes all operators previously suggested to be important for capturing the long-distance behavior of the present disordered models, namely operators involving 1- and 2-replica terms. As will be shown further below, it also reduces to the leading results of perturbative RG analyses near the upper critical dimension, $d_{uc} = 6$, near the lower critical dimension for ferromagnetism when $N > 1$, $d = 4$, and when the number of components N becomes infinite. One of its main advantages is that it provides a unified framework to describe models in any spatial dimension d and for any number N of field components. As such, it guarantees a consistent interpolation of all known results in the whole (N, d) plane, in addition to allowing the study of genuine nonperturbative phenomena. If more accuracy is needed, the truncation scheme proposed in III-B gives a systematic means to refine the description, by including *e.g.* the third cumulant or a more detailed account of the momentum dependence of the 1-PI vertices.

In the following, we more specifically focus on the random field $O(N)$ model.

IV. RANDOM FIELD MODEL

A. Scaling dimensions near a zero-temperature fixed point

For the RFIM, it has been proposed^{64,65}, and convincingly supported by numerical and experimental results^{6,40,66}, that the fixed point controlling the critical behavior associated with the transition between a high-temperature - or large-disorder strength - disordered (paramagnetic) phase and a low-temperature - or small-disorder strength - ordered (ferromagnetic) phase is at zero temperature (see Figure 1). The existence of

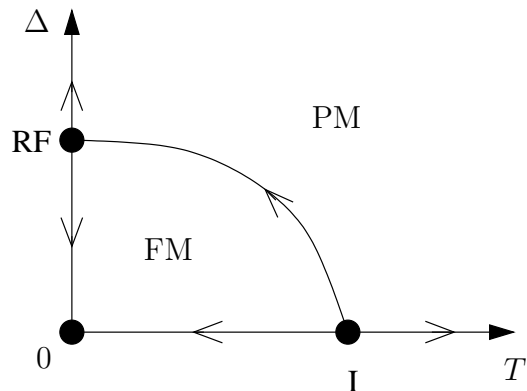


FIG. 1: Schematic phase diagram of the RFIM in the disorder strength Δ - temperature T plane above the lower critical dimension $d_{lc} = 2$ (temperature can be introduced at the bare level through the Boltzmann weight). At low disorder and low temperature, the system is ferromagnetic, and it is paramagnetic otherwise. The arrows describe how the renormalized parameters evolve under the RG flow at long distance, and I and RF denote the critical fixed points of the pure and random-field Ising models, respectively.

such a zero-temperature fixed point around which temperature is dangerously irrelevant leads to a somewhat anomalous scaling at the critical point.^{64,65} The two independent critical exponents characterizing the scaling behavior of the pure Ising model should *a priori* be supplemented by an additional exponent θ describing the vanishing of the (renormalized) temperature as the fixed point is approached. This exponent θ leads to a modification of the so-called hyperscaling relation, which becomes $2 - \alpha = (d - \theta)\nu$ where the critical exponents α and ν have their usual meaning, and to a new scaling of the correlation functions. In particular, the so-called “connected” and “disconnected” components of the pair correlation function (or 2-point Green function) behave at the critical point as:

$$G_c(q) = \overline{\langle \chi(-q)\chi(q) \rangle} - \overline{\langle \chi(-q) \rangle} \overline{\langle \chi(q) \rangle} \sim q^{-(2-\eta)} \quad (52)$$

$$G_d(q) = \overline{\langle \chi(-q) \rangle} \overline{\langle \chi(q) \rangle} \sim q^{-(4-\bar{\eta})} \quad (53)$$

where η is the usual anomalous dimension of the field and $\bar{\eta}$ is related to the temperature exponent θ through

$$\bar{\eta} = 2 - \theta + \eta. \quad (54)$$

Above the upper critical dimension $d_{uc} = 6$, the exponents take their classical, mean-field values, $\eta = 0, \alpha = 0, \nu = 1/2$, and $\theta = 2$, leading to $\bar{\eta} = 0$. The dimensional reduction property leads to a constant shift of dimension, $d \rightarrow d - 2$, *i.e.*, to $\theta = 2$ and $\bar{\eta} = \eta$, all exponents being in addition given by those of the pure model in dimension $d - 2$. Whether the scaling behavior around the critical point is described by 3 independent exponents, or only 2, has been a long-time issue, with suggestions that an additional relation applies, $\theta = 2 - \eta$ or equivalently $\bar{\eta} = 2\eta$.⁶⁷ We shall address and answer this question in the following paper.³²

To search for a zero-temperature fixed point, it is convenient to introduce a renormalized temperature. Actually, one could add an explicit temperature T in the Landau-Ginzburg-Wilson description of the model considered here: multiplying the argument of the exponential in the partition function, Eq. (4), by a factor T^{-1} to make the correspondence with the Boltzmann factor of Statistical Physics leads to a bare replicated action in Eqs. (6) and (10) in which the 1-replica part, including the kinetic term, is multiplied by a factor T^{-1} , the 2-replica part by T^{-2} , etc. Generally speaking, one can use this temperature T as a book keeping device to sort the orders in the expansions in number of free replica sums. As a result for instance, the modified propagator $\hat{P}_k^{[0]}[\phi_1]$ is proportional to T whereas $\tilde{P}_k^{[0]}[\phi_1, \phi_2]$ is independent of T . One can use this book keeping trick to devise ways to define a renormalized temperature at running scale k , T_k , which reduces to the “bare” temperature T at the microscopic scale $k = \Lambda$. To this end, we first define the renormalized disorder strength at scale k , $\Delta_{m,k}$, as

$$\Delta_{m,k} = \Delta_k(\phi_1 = \phi_{m,k}, \phi_2 = \phi_{m,k}), \quad (55)$$

where as before $\phi_{m,k}$ is a field configuration that minimizes the (1-replica) potential $U_k(\phi)$, and $\Delta_k(\phi_1, \phi_2)$ is the second cumulant of the renormalized effective random field defined as in Eq. (29), namely,

$$\Delta_k(\phi_1, \phi_2) = V_k^{(11)}(\phi_1, \phi_2). \quad (56)$$

In the present truncation, the second cumulant is only considered for homogeneous field configurations and $\Gamma_{k,2}^{(11)}$ reduces to $V_k^{(11)}$ with the same notations for partial derivatives as in Eqs. (26, 27) (*e.g.*, $V_k^{(11)}(\phi_1, \phi_2) = \partial_{\phi_1} \partial_{\phi_2} V_k(\phi_1, \phi_2)$). At the microscopic scale Λ , $\Delta_{m,k}$ reduces to Δ_Λ/T^2 where Δ_Λ is the bare variance of the random field and the factor T^{-2} comes for reasons just explained above.

A running temperature can now be defined by

$$T_k = \frac{Z_{m,k}(k/\Lambda)^2}{(\Delta_{m,k}/\Delta_\Lambda)^2}. \quad (57)$$

One checks that since $Z_{m,\Lambda} = T^{-1}$ (see Eq. (10) and discussion above), T_k indeed reduces to T when $k = \Lambda$. An associated running exponent is obtained from

$$\theta_k = k \partial_k \ln T_k. \quad (58)$$

By using the definition of η_k , one may alternatively introduce a running exponent $\bar{\eta}_k = 2 - \theta_k + \eta_k$, which converges to the critical exponent $\bar{\eta}$ defined in Eqs. (53, 54) if the relevant fixed point is reached, and compute it from the equation

$$\bar{\eta}_k - 2\eta_k = k \partial_k \Delta_{m,k}. \quad (59)$$

On top of the usual scaling dimensions, $U_k, V_k \sim k^d$ and $\phi \sim (Z_{m,k}^{-1} k^{d-2})^{1/2}$, one can use the running temperature to define dimensionless quantities (denoted by lower-case letters) suitable for looking for a zero-temperature fixed point:

$$\phi = \left(\frac{k^{d-2}}{Z_{m,k} T_k} \right)^{1/2} \varphi \quad (60a)$$

$$U_k(\phi_1) = \frac{k^d}{T_k} u_k(\varphi_1) \quad (60b)$$

$$V_k(\phi_1, \phi_2) = \frac{k^d}{T_k^2} v_k(\varphi_1, \varphi_2) \quad (60c)$$

$$\Delta_k(\phi_1, \phi_2) = \frac{Z_{m,k} k^2}{T_k} \delta_k(\varphi_1, \varphi_2), \quad (60d)$$

with $\delta_k(\varphi_1, \varphi_2) = v_k^{(11)}(\varphi_1, \varphi_2)$. Note that with the definitions of $\Delta_{m,k}$ and T_k , $\delta_{m,k} \equiv \delta_k(\varphi_{m,k}, \varphi_{m,k})$ is constant along the RG flow and equal to its initial value Δ_Λ/Λ^2 (in practice, and since we are not interested here in making a precise connection to the microscopic scale, we will set $\delta_{m,k} = 1$).

B. Scaled form of the exact RG equations for the RFIM

With the use of the above defined dimensionless renormalized quantities, the flow equations can be expressed in a scaled form. Specifically, one can recast Eqs.(47) and (48) for $N = 1$ in the form

$$\begin{aligned} \partial_t u_k(\varphi) = & -(d - 2 + \bar{\eta}_k - \eta_k) u_k(\varphi) \\ & + \frac{1}{2} (d - 4 + \bar{\eta}_k) \varphi u'_k(\varphi) \\ & + 2v_d \left\{ l_1^{(d)}(u''_k(\varphi)) \delta_k(\varphi, \varphi) + T_k l_0^{(d)}(u''_k(\varphi)) \right\} \end{aligned} \quad (61)$$

$$\begin{aligned}
\partial_t v_k(\varphi_1, \varphi_2) = & -(d-4+2\bar{\eta}_k-2\eta_k)v_k(\varphi_1, \varphi_2) + \\
& \frac{1}{2}(d-4+\bar{\eta}_k)(\varphi_1\partial_{\varphi_1}+\varphi_2\partial_{\varphi_2})v_k(\varphi_1, \varphi_2) \\
& - 2v_d \left\{ l_{1,1}^{(d)}(u_k''(\varphi_1), u_k''(\varphi_2))\delta_k(\varphi_1, \varphi_2)^2 \right. \\
& + l_2^{(d)}(u_k''(\varphi_1))\delta_k(\varphi_1, \varphi_1)v_k^{(20)}(\varphi_1, \varphi_2) \\
& + l_2^{(d)}(u_k''(\varphi_2))\delta_k(\varphi_2, \varphi_2)v_k^{(02)}(\varphi_1, \varphi_2) \\
& + T_k[l_1^{(d)}(u_k''(\varphi_1))v_k^{(20)}(\varphi_1, \varphi_2) \\
& \left. + l_1^{(d)}(u_k''(\varphi_2))v_k^{(02)}(\varphi_1, \varphi_2)] \right\}
\end{aligned} \tag{62}$$

where ∂_t is a derivative with respect to $t = \ln(k/\Lambda)$, a prime denotes a derivative with respect to the field (when only one argument is present), $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$, and we recall that $\delta_k(\varphi_1, \varphi_2) = v_k^{(11)}(\varphi_1, \varphi_2)$; $l_n^{(d)}(w)$ and $l_{n_1, n_2}^{(d)}(w_1, w_2)$ are the “dimensionless threshold functions” defined from the infrared cutoff function, Eq.(13), as^{36,51}

$$l_n^{(d)}(w) = -\frac{1}{2}(n+\delta_{n,0}) \int_0^\infty dy y^{d/2} \frac{\eta_k r(y) + 2yr'(y)}{(p(y)+w)^{n+1}}, \tag{63}$$

$$\begin{aligned}
l_{n_1, n_2}^{(d)}(w_1, w_2) = & -\frac{1}{2} \int_0^\infty dy y^{d/2} (\eta_k r(y) + 2yr'(y)) \\
& \frac{1}{(p(y)+w_1)^{n_1}(p(y)+w_2)^{n_2}} \left(\frac{n_1}{p(y)+w_1} + \frac{n_2}{p(y)+w_2} \right),
\end{aligned} \tag{64}$$

with $p(y) = y(1+r(y))$ and $y = q^2/k^2$. The properties of these threshold functions, whose detailed behavior depends on the choice of the infrared cut-off function $r(y)$, have been extensively discussed.^{36,51} They decay rapidly when $w \gg 1$, which, since $u_k''(\varphi) = U_k''(\phi)/(Z_k k^2)$ is the square of a renormalized mass, ensures that only modes with mass smaller than k contribute to the flow in Eqs. (61) and (62). As an illustration, the use of the so-called “optimized” cut-off function $r(y) = y^{-1}(1-y)\Theta(1-y)$,⁵² leads to explicit expressions, namely,

$$\begin{aligned}
l_{n_1, n_2}^{(d)}(w_1, w_2) = & \frac{2}{d} \left(1 - \frac{\eta_k}{d+2} \right) \frac{1}{(1+w_1)^{n_1}(1+w_2)^{n_2}} \\
& \times \left(\frac{n_1}{1+w_1} + \frac{n_2}{1+w_2} \right).
\end{aligned} \tag{65}$$

The threshold functions essentially encode the nonperturbative effects beyond the standard one-loop approximation. Note that, although not shown in the notation, the threshold functions explicitly depend on the scale k via the running exponent η_k .

The above flow equations for $u_k(\varphi_1)$ and $v_k(\varphi_1, \varphi_2)$ are supplemented by equations for η_k and $\bar{\eta}_k$, i.e., for $Z_{m,k}$ and T_k or $\Delta_{m,k}$. (Note that the equation for $\bar{\eta}_k$ is actually redundant as it is a consequence of the other equations; it is nonetheless convenient to introduce and

use it.) The flow equation for $Z_{m,k}$ follows from Eq.(50) and one finds:

$$\begin{aligned}
\eta_k = & \frac{4v_d}{d} \left\{ 4m_{3,2}^{(d)}(u_{m,k}'', u_{m,k}'')u_{m,k}''' - \right. \\
& 2m_{2,2}^{(d)}(u_{m,k}'', u_{m,k}'')u_{m,k}'''\delta_{m,k}' - \\
& \left. + T_k m_{2,2}^{(d)}(u_{m,k}'', u_{m,k}'')u_{m,k}''' \right\}
\end{aligned} \tag{66}$$

where we have used the short-hand notation $\delta_k'(\varphi) \equiv \partial_\varphi \delta_k(\varphi, \varphi) = \delta_k^{(10)}(\varphi, \varphi) + \delta_k^{(01)}(\varphi, \varphi)$ and the subscript “ m, k ” indicates that the functions are evaluated for fields equal to $\varphi_{m,k}$; we have also introduced the additional (dimensionless) threshold function

$$\begin{aligned}
m_{n_1, n_2}^{(d)}(w_1, w_2) = & -\frac{1}{2} \int_0^\infty dy y^{d/2} (1+r(y) + yr'(y)) \\
& \frac{1}{(p(y)+w_1)^{n_1}(p(y)+w_2)^{n_2}} \left\{ (1+r(y) + yr'(y)) \right. \\
& (\eta_k r(y) + 2yr'(y)) \left(\frac{n_1}{p(y)+w_1} + \frac{n_2}{p(y)+w_2} \right) - \\
& \left. 2\eta_k(r(y) + yr'(y)) - 4y(2r'(y) + yr''(y)) \right\},
\end{aligned} \tag{67}$$

whose properties are discussed in Ref. [36,51]. For instance, with the “optimized” regulator introduced above,⁵² one finds that

$$m_{n_1, n_2}^{(d)}(w_1, w_2) = \frac{1}{(1+w_1)^{n_1}(1+w_2)^{n_2}}. \tag{68}$$

Finally the flow equation for $\Delta_{m,k}$ (or equivalently the flow of the constraint $\delta_{m,k} = 1$ discussed below Eq. (60)) leads to the following equation:

$$\begin{aligned}
2\eta_k - \bar{\eta}_k = & 2v_d \left\{ l_4^{(d)}(u_{m,k}'', u_{m,k}'')u_{m,k}''' - 4l_3^{(d)}(u_{m,k}'')u_{m,k}'''\delta_{m,k}' \right. \\
& + l_2^{(d)}(u_{m,k}'')(\delta_{m,k}'' + \frac{3}{2}\delta_{m,k}'^2 - \frac{u_{m,k}'''}{u_{m,k}''} - \frac{1}{4}\Sigma_{m,k}) \\
& + l_1^{(d)}(u_{m,k}'')\frac{\delta_{m,k}'^2}{u_{m,k}''} - T_k \left[l_2^{(d)}(u_{m,k}'')u_{m,k}'''\delta_{m,k}' \right. \\
& \left. - l_1^{(d)}(u_{m,k}'')(\frac{1}{2}\delta_{m,k}'' - \frac{u_{m,k}'''}{u_{m,k}''}\delta_{m,k}' + \frac{1}{2}\tilde{\Sigma}_{m,k}) \right] \left. \right\},
\end{aligned} \tag{69}$$

where, as before, $\delta_k'(\varphi) \equiv \partial_\varphi \delta_k(\varphi, \varphi)$ and similarly for $\delta_k''(\varphi)$, and we have introduced

$$\Sigma_k(\varphi_1) = \lim_{\varphi_2 \rightarrow \varphi_1} (\partial_{\varphi_1} - \partial_{\varphi_2})^2 (\delta_k(\varphi_1, \varphi_2) - \delta_k(\varphi_1, \varphi_1))^2 \tag{70}$$

and

$$\tilde{\Sigma}_k(\varphi_1) = \lim_{\varphi_2 \rightarrow \varphi_1} (\partial_{\varphi_1} - \partial_{\varphi_2})^2 \delta_k(\varphi_1, \varphi_2). \tag{71}$$

All other notations are as before.

Before extending the results to the RFO(N)M, we point out important features of the above equations. First, we have kept terms proportional to T_k but, provided one reaches a fixed point with an exponent $\theta = \theta_{k \rightarrow 0} > 0$ where temperature is thus irrelevant, those terms are subdominant in the scaling region $k \rightarrow 0$. In particular, the fixed point is attained by following the flow with an initial temperature T equal to zero.

Secondly, “anomalous” terms, $\Sigma_{m,k}$ and $T_k \tilde{\Sigma}_{m,k}$, appear in the expression of $2\eta_k - \bar{\eta}_k$. As can be inferred from Eqs. (70) and (71), $\Sigma_{m,k}$ can only differ from zero, and $\tilde{\Sigma}_{m,k}$ become infinite, when a non-analyticity (a “cusp”) in $(\varphi_1 - \varphi_2)$ appears in the (dimensionless) renormalized disorder function $\delta_k(\varphi_1, \varphi_2)$ when $\varphi_2 \rightarrow \varphi_1$ (and both go to $\varphi_{m,k}$). If $\delta_k(\varphi_1, \varphi_2)$ is analytic, no signature of such anomalous behavior is found. (We have implicitly assumed that no stronger nonanalyticity appears, which means that a fixed point can be reached and that the theory is renormalizable; this has to be checked in actual computations.) We shall come back in more detail to these two important aspects of the NP-FRG approach in the following paper.³² Finally, one may notice that because of the $Z_2 \equiv O(1)$ symmetry, the potential u_k is an even function of φ and because of the additional permutation symmetry, $v_k(\varphi_1, \varphi_2) = v_k(\varphi_2, \varphi_1) = v_k(-\varphi_1, -\varphi_2) = v_k(-\varphi_2, -\varphi_1)$.

C. Generalization to the RFO(N)M

The preceding treatment can be extended to the RFO(N)M. The variable $\rho = \frac{1}{2}|\phi|^2$ is written in terms of a dimensionless variable, $\tilde{\rho} = k^{d-2} T_k^{-1} Z_{m,k}^{-1} \rho$, where the tilde will be dropped in the following when no confusion is possible between dimensionless and dimensionful quantities. The variable $z = \phi_1 \cdot \phi_2 / (2\sqrt{\rho_1 \rho_2})$ is already dimensionless.

For the 1-replica second-order tensors (in N -vector components) evaluated for a uniform field configuration, *e.g.*, for $\hat{P}_k^{[0]}(q^2; \phi_1)$ or for $\Delta_k(\phi_1, \phi_1) \equiv V_k^{(11)}(\phi_1, \phi_1)$, the $O(N)$ symmetry reduces the number of terms to a “longitudinal” component (corresponding to the massive mode, see Eq. (49)) and $N - 1$ identical “transverse” components (corresponding to the Goldstone modes, see Eq. (49)). We therefore introduce

$$\delta_k^{\mu\nu}(\rho, \rho, z = 1) = \delta_{\mu\nu} [\delta_{\mu 1} \delta_{k,L}(\rho) + (1 - \delta_{\mu 1}) \delta_{k,T}(\rho)], \quad (72)$$

with

$$\delta_{k,L}(\rho) = 2\rho \partial_{\rho_1} \partial_{\rho_2} v(\rho_1, \rho_2, z = 1)|_{\rho_1 = \rho_2 = \rho}, \quad (73)$$

$$\delta_{k,T}(\rho) = \frac{1}{2\rho} \partial_z v(\rho, \rho, z)|_{z=1}, \quad (74)$$

and we define the longitudinal, $w_{k,L}(\rho)$, and transverse, $w_{k,T}(\rho)$, masses as

$$w_{k,L}(\rho) = u'_k(\rho) + 2\rho u''_k(\rho), \quad (75)$$

$$w_{k,T}(\rho) = u'_k(\rho), \quad (76)$$

where a prime now denotes a derivative with respect to ρ .

The renormalized disorder strength at the running scale k can be characterized, *e.g.*, through the transverse component, $\Delta_{k,T}(\rho, \rho, z = 1)$, evaluated for $\rho = \rho_{m,k} = \frac{1}{2}|\phi_{m,k}|^2$, and T_k is introduced accordingly. Expressing the $O(N)$ symmetry in the 2-replica second-order tensors is a little more tedious, but nonetheless straightforward.

The resulting flow equations in scaled form read (where for ease of notation we drop the subscript k in the right-hand sides, *i.e.*, up to a sign, the beta functions, for all quantities but T_k and also drop the argument of $v(\rho_1, \rho_2, z)$):

$$\begin{aligned} \partial_t u_k(\rho) = & -(d - 2 + \bar{\eta} - \eta)u(\rho) + (d - 4 + \bar{\eta})\rho u'(\rho) \\ & + 2v_d \left\{ (N - 1)l_1^{(d)}(w_T(\rho))\delta_T(\rho) + l_1^{(d)}(w_L(\rho))\delta_L(\rho) \right\} \\ & + 2T_k v_d \left\{ (N - 1)l_0^{(d)}(w_T(\rho)) + l_0^{(d)}(w_L(\rho)) \right\}, \end{aligned} \quad (77)$$

$$\begin{aligned} \partial_t v_k(\rho_1, \rho_2, z) = & -(d - 4 + 2\bar{\eta} - 2\eta)v + (d - 4 + \bar{\eta}) \\ & (\rho_1 \partial_{\rho_1} + \rho_2 \partial_{\rho_2})v - \frac{v_d}{4\rho_1 \rho_2} \left\{ (N - 1) \left[4\rho_2 l_2^d(w_T(\rho_1)) \right. \right. \\ & \delta_T(\rho_1)(2\rho_1 \partial_{\rho_1} v - z \partial_z v) + l_{1,1}^{(d)}(w_T(\rho_1), w_T(\rho_2))(\partial_z v)^2 \\ & + (1 - z^2) \left[4\rho_2 l_2^{(d)}(w_T(\rho_1))\delta_T(\rho_1)\partial_z^2 v + \right. \\ & 8\rho_2^2 l_{1,1}^{(d)}(w_T(\rho_1), w_L(\rho_2))(\partial_{\rho_2} \partial_z v)^2 - l_{1,1}^{(d)}(w_T(\rho_1), w_T(\rho_2)) \\ & \left. \left. ((\partial_z v)^2 + 2z \partial_z v \partial_z^2 v - (1 - z^2)(\partial_z^2 v)^2) \right] \right. \\ & + 8\rho_1 \rho_2 \left[l_2^{(d)}(w_L(\rho_1))\delta_L(\rho_1)(\partial_{\rho_1} v + 2\rho_1 \partial_{\rho_1}^2 v) \right. \\ & + 2\rho_1 \rho_2 l_{1,1}^{(d)}(w_L(\rho_1), w_L(\rho_2))(\partial_{\rho_1} \partial_{\rho_2} v)^2 \left. \right] + \text{perm}(12) \left\{ \right. \\ & - T_k \frac{v_d}{\rho_1 \rho_2} \left\{ (N - 1)\rho_2 l_1^d(w_T(\rho_1))(2\rho_1 \partial_{\rho_1} v - z \partial_z v) + \right. \\ & (1 - z^2)\rho_2 l_1^d(w_T(\rho_1))\partial_z^2 v + 2\rho_1 \rho_2 l_1^d(w_L(\rho_1)) \\ & \left. \left. (\partial_{\rho_1} v + 2\rho_1 \partial_{\rho_1}^2 v) + \text{perm}(12) \right\} \right\}, \end{aligned} \quad (78)$$

$$\begin{aligned} \eta_k = & \frac{v_d}{d} \left\{ 8 \left[m_{2,3}^{(d)}(w_L(\rho_m), 0) + m_{3,2}^{(d)}(w_L(\rho_m), 0) \right] \right. \\ & \delta_T(\rho_m) \frac{w_L(\rho_m)^2}{\rho_m} + 8m_{3,1}^{(d)}(w_L(\rho_m), 0) \\ & \left. \times w_L(\rho_m) \left[\delta_T(\rho_m) - \delta_L(\rho_m) \right] \right\} \end{aligned} \quad (79)$$

$$\begin{aligned}
2\eta_k - \bar{\eta}_k = & \frac{2v_d}{\rho_m u''(\rho_m)} \left\{ (N-1)\rho_m l_1^d(0)\delta'_T(\rho_m)^2 + \right. \\
& l_2^d(0)u''(\rho_m) + l_2^{(d)}(w_L(\rho_m))\delta_L(\rho_m) \left[(1 + 2\rho_m\delta'_T(\rho_m) \right. \\
& \left. + 2\rho_m^2\delta''_T(\rho_m))u''(\rho_m) - 2\rho_m^2u'''(\rho_m)\delta'_T(\rho_m) \right] \\
& \left. - 2l_{1,1}^{(d)}(0, w_L(\rho_m))u''(\rho_m) \left[1 + \rho_m\delta'_T(\rho_m) \right]^2 \right. \\
& \left. + \rho_m l_1^{(d)}(w_L(\rho_m))\delta'_T(\rho_m)\delta'_L(\rho_m) \right\} + \dots, \tag{80}
\end{aligned}$$

where all symbols have the same meaning as in the previous equations and, by construction, $w_L(\rho_m) = 2\rho_m u''(\rho_m)$, $w_T(\rho_m) = 0$, and $\delta_T(\rho_m) = 1$. Note that in the last two equations, we have omitted for simplicity the (subdominant) terms involving T_k in the beta functions and that in Eq. (80), the dots denote “anomalous” terms which generalize those found for the RFIM (see Eq. (69)) and vanish when the function $v_k(\rho_1, \rho_2, z)$ is analytic in all its arguments; their expression is lengthy and will be discussed in the companion paper.³²

When $N = 1$ and $z = \pm 1$, Eqs. (77) and (78) reduce to the previous equations for the RFIM, Eqs. (61) and (62), expressed with ρ as variable instead of ϕ : $v_k(\rho_1, \rho_2, z = +1)$ is equal to $v_k(\varphi_1, \varphi_2)$ for $\varphi_1\varphi_2 > 0$ and $v_k(\rho_1, \rho_2, z = -1)$ is equal to $v_k(\varphi_1, \varphi_2)$ for $\varphi_1\varphi_2 < 0$; $\delta_{k,L}(\rho) \equiv \delta_k(\varphi)$ and $w_{k,L}(\rho) \equiv u''(\varphi)$.⁷⁵ Finally, the comments made about the important features of the flow equations for the RFIM carry over to the equations for the RFO(N)M.

D. Application to related disordered models

Even though we have chosen to more specifically focus on the random field model, it is worth sketching at this point the relevance of the NP-FRG equations derived in this section to other disordered systems. (As stressed already several times, we exclude spin glass ordering from our considerations.)

The flow equations obtained for the RFO(N)M, Eqs. (77-79), directly apply to the RAO(N)M for describing the long-distance physics associated with ferromagnetic ordering. The putative fixed points are also expected to be at zero temperature, so that similar scaling dimensions need be introduced. The specificity of the random anisotropy model comes in the initial conditions (see section II-A) and in the additional symmetry of the 2-replica potential, namely, $v_k(\rho_1, \rho_2, z) = v_k(\rho_1, \rho_2, -z)$.

Similarly, the flow equations for the RFIM, Eqs. (61,62,66), can be applied to the random elastic model. In this case, one can check that, owing to the statistical tilt symmetry, $u'_k(\varphi) \equiv 0$ and $\eta_k \equiv 0$ while $v_k(\varphi_1, \varphi_2) \equiv v_k(\varphi_1 - \varphi_2)$. After introducing the variable $y = \varphi_1 - \varphi_2$ and dropping the temperature, Eq. (62),

can be rewritten as

$$\begin{aligned}
-\partial_t v_k(y) = & (d-4+2\bar{\eta}_k)v_k(y) - \frac{1}{2}(d-4+\bar{\eta}_k)yv'_k(y) \\
& + 2v_d l_2^{(d)}(0) [v''_k(y) - 2v''_k(0)] v''_k(y), \tag{81}
\end{aligned}$$

where a prime denotes a derivative with respect to y . The roughness exponent is defined through $\zeta = -(d-4+\bar{\eta})/2$, and one can then see that the above equation reduces to the one-loop FRG equation for a disordered elastic medium.^{28,44} Going beyond this level of description requires to consider the next orders of the truncation scheme, in particular to include the 3-replica potential and apply the next order of the derivative expansion for the 2-replica effective average action.

Finally, Eqs. (61,62,66) can be used in the case of the random temperature model with an appropriate account of the symmetry: $u_k \equiv u_k(\rho)$, $v_k \equiv v_k(\rho_1, \rho_2)$, with $\rho = \varphi^2/2$. However, the scaling dimensions introduced to search for a zero-temperature fixed point are not appropriate in the present case where one anticipates a fixed point at a nonzero temperature (for a preliminary non-perturbative treatment, see Ref. [68]).

V. RECOVERING THE PERTURBATIVE RESULTS

A. Analysis of the NP-FRG equations near $d = 6$ and for $N \rightarrow \infty$

For ease of notation, we only consider the RFIM, but a similar analysis holds for the RFO(N)M. It is easy to check that the flow equations, Eqs. (61, 62,66,69), admit for fixed-point solution the Gaussian fixed point characterized by $\eta_*^{(G)} = \bar{\eta}_*^{(G)} = 0$, $u_*^{(G)}(\varphi) = 2v_d l_1^{(d)}(0)/(d-2)$, and $\delta_*^{(G)}(\varphi_1, \varphi_2) = 1$. The Gaussian fixed point is once unstable for dimensions larger than 6, but the coupling constant associated with the φ^4 -term in $u(\varphi)$ also becomes relevant for dimensions less than 6 so that the Gaussian fixed point becomes unstable for $d < 6$, as already well known.

The first order in $\epsilon = 6 - d$ can be derived by a direct expansion of the fixed-point solution, with $u_*(\varphi) = u_*^{(G)} + \epsilon u_1(\varphi)$, $\delta_*(\varphi_1, \varphi_2) = 1 + \epsilon \delta_1(\varphi_1, \varphi_2)$. One easily finds that at this order one still has $\eta_* = \bar{\eta}_* = 0$. After inserting these results in Eqs. (61,62), deriving the equation for v_k with respect to φ_1 and φ_2 , and setting the left-hand sides to zero, one obtains the following equations for $u_1(\varphi)$ and $\delta_1(\varphi_1, \varphi_2)$:

$$\begin{aligned}
0 = & 4u_1(\varphi) - \varphi u'_1(\varphi) - \frac{v_6}{2} \left[l_1^{(6)}(0) - 4l_2^{(6)}(0)u''_1(\varphi) \right. \\
& \left. + 4l_1^{(6)}(0)\delta_1(\varphi, \varphi) \right], \tag{82}
\end{aligned}$$

$$0 = (\varphi_1 \partial_{\varphi_1} + \varphi_2 \partial_{\varphi_2}) \delta_1(\varphi_1, \varphi_2) - 2v_6 l_2^{(6)}(0) (\partial_{\varphi_1} + \partial_{\varphi_2})^2 \delta_1(\varphi_1, \varphi_2). \quad (83)$$

By introducing the variables $x = (\varphi_1 + \varphi_2)/2$ and $y = (\varphi_1 - \varphi_2)/2$, the latter equation can be rewritten as

$$(x \partial_x + y \partial_y) \delta_1(x, y) = 2v_6 l_2^{(6)}(0) \partial_x^2 \delta_1(x, y). \quad (84)$$

The symmetry of $v(\varphi_1, \varphi_2)$ with respect to the exchange of φ_1 and φ_2 and to changes of sign of φ_1 and φ_2 (see IV-B) translates into the fact that δ_1 is an even function of x and y . Provided one requires that $\delta_1(x, y \rightarrow 0)$ is finite (which is needed for a well defined renormalizable theory), the only acceptable solution satisfying this property is a constant; due to the constraint $\delta_{m,k} \equiv \delta_k(\varphi_{m,k}) = 1$, it is equal to zero, *i.e.*, $\delta_1(x, y) = 0$. In addition to the now unstable Gaussian fixed point, Eq. (82) has then for solution $u_1(\varphi) = (\lambda_{1*}/8)(\varphi^2 - \varphi_{m*}^2)^2 + \text{constant}$ with $\varphi_{m*}^2 = 6v_6 l_2^{(6)}(0)$. One also finds $\lambda_{1*} = (36v_6 l_3^{(6)}(0))^{-1}$, so that, up to irrelevant constant factors, the solution corresponds to the fixed point of the pure Ising model (no random field) at first order in $\epsilon = 4 - d$. The fixed point is found once unstable and the associated exponents, *e.g.*, $\nu = \frac{1}{2} + \frac{\epsilon}{12}$, satisfy the $d \rightarrow d - 2$ dimensional reduction.

Equivalently, one can make a more direct connection to standard perturbation analysis by reframing the above results in a double expansion in ϵ and in the φ^4 coupling constant defined through $\lambda_k = u_k''''(\varphi_{m,k})$. Introducing as before $\rho_{m,k} = (1/2)\varphi_{m,k}^2$, one obtains from Eqs. (77-80) that $\eta, \bar{\eta} = O(\lambda^2)$, $\delta = 1 + O(\lambda^2)$ and

$$\partial_t \lambda_k = -\epsilon \lambda_k + 36v_d l_3^{(d)}(0) \lambda_k^2 + O(\lambda_k^3, T_k \lambda_k^2), \quad (85)$$

$$\begin{aligned} \partial_t \rho_{m,k} = & -(2 - \epsilon) \rho_{m,k} + 6v_d [l_2^{(d)}(0) - 4l_3^{(d)}(0) \lambda_k \rho_{m,k}] \\ & + O(\lambda_k^3, T_k \lambda_k^2), \end{aligned} \quad (86)$$

where we have used the Taylor expansion of the threshold functions for small arguments. (The fixed-point solution of Eqs. (85,86) is of course equal to that obtained above with $\lambda_* = \epsilon \lambda_{1*}$ and $\rho_{m*} = \varphi_{m*}^2/2$.) Again, up to irrelevant factors, this gives back the one-loop perturbative result for the pure Ising model obtained in a weak-coupling expansion in $d = 4 - \epsilon$.

The above result is derived through an expansion in a single coupling constant, λ_k , associated to the 1-replica part of the effective action. It has been argued by Brezin and De Dominicis^{69,70} that one should consider instead an expansion involving all φ^4 coupling constants associated with multiple replicas. In the present formalism, we can perform a more careful analysis using the φ^4 coupling constants associated with the 2-replica part of the effective action, coupling constants that are considered as potentially relevant in Refs. [69,70]. We find that this does not change the conclusion and, as previously obtained in Ref. [71], that the fixed point corresponding to

dimensional reduction is still once unstable at first order in ϵ . This is discussed in more detail in Appendix A.

The above analysis is extended to the $O(N)$ version in a straightforward way. The property that the perturbative result at first order in $\epsilon = 6 - d$ is recovered within our nonperturbative approximation scheme is actually a consequence of the one-loop-like structure of the exact flow equation for the effective average action, Eq. (16). For the very same reason, the large N limit can also be easily recovered.

Rescaling the variables as $\rho \rightarrow N\rho$, $z \rightarrow z$ and the potentials as $u \rightarrow Nu$, $v \rightarrow Nv$, and retaining only the dominant terms when $N \rightarrow \infty$, one finds that $\eta = O(1/N)$, $\bar{\eta} = O(1/N)$ and that the “longitudinal” contributions drop out from the RG flow equations. As a consequence, Eqs. (77) and (78) can be recast as

$$\begin{aligned} \partial_t u_k(\rho) = & -(d-2)u_k(\rho) + (d-4)\rho u_k'(\rho) \\ & + 2v_d l_1^{(d)}(u_k'(\rho)) \delta_{k,T}(\rho), \end{aligned} \quad (87)$$

$$\begin{aligned} \partial_t \delta_{k,T}(\rho_1, \rho_2, z) = & (d-4)(\rho_1 \partial_{\rho_1} + \rho_2 \partial_{\rho_2}) \delta_{k,T}(\rho_1, \rho_2, z) - \\ & 2v_d \left\{ \frac{l_{11}^{(d)}(u_k'(\rho_1), u_k'(\rho_2))}{\sqrt{\rho_1 \rho_2}} \delta_{k,T}(\rho_1, \rho_2, z) \partial_z \delta_{k,T}(\rho_1, \rho_2, z) \right. \\ & + \frac{l_2^{(d)}(u_k'(\rho_1))}{2\rho_1} \delta_{k,T}(\rho_1) (2\rho_1 \partial_{\rho_1} - z \partial_z) \delta_{k,T}(\rho_1, \rho_2, z) \\ & \left. + \frac{l_2^{(d)}(u_k'(\rho_2))}{2\rho_2} \delta_{k,T}(\rho_2) (2\rho_2 \partial_{\rho_2} - z \partial_z) \delta_{k,T}(\rho_1, \rho_2, z) \right\}, \end{aligned} \quad (88)$$

where we have defined a generalized “transverse” disorder cumulant $\delta_{k,T}(\rho_1, \rho_2, z)$ via an extension of Eq.(74), namely,

$$\delta_{k,T}(\rho_1, \rho_2, z) = \frac{1}{2\sqrt{\rho_1 \rho_2}} \partial_z v_k(\rho_1, \rho_2, z), \quad (89)$$

which reduces to $\delta_{k,T}(\rho)$ when $\rho_1 = \rho_2 = \rho$ and $z = 1$. Eq. (88) is obtained by deriving the flow equation for $v_k(\rho_1, \rho_2, z)$.

If one starts the flow equations with an initial condition $v_\Lambda(\rho_1, \rho_2, z) = 2\sqrt{\rho_1 \rho_2} z$ (corresponding to $\delta_{\Lambda,T} = 1$), the beta function is identically zero and one therefore finds that the solution of Eq. (88) at all scales remains $\delta_{k,T}(\rho_1, \rho_2, z) = 1$.⁷⁶ The resulting equation for the 1-replica potential is then very similar to its counterpart for the pure $O(N)$ model with $N \rightarrow \infty$ limit in dimension $d-2$ (the flow equation is then simply given by the LPA³⁶).

To see more explicitly the connection, one can follow the flow of the φ^4 coupling constant $\lambda_k = u_k''(\rho_{m,k})$ as well as that of $\rho_{m,k}$ which, we recall, satisfies $u_k'(\rho_{m,k}) = 0$ and is akin to a (dimensionless) order parameter at the running scale k . One finds

$$\begin{aligned} \partial_t \lambda_k = & -(6-d)\lambda_k + 4v_d l_3^{(d)}(0) \lambda_k^2 \\ & + [(d-4)\rho_{m,k} - 2v_d l_2^{(d)}(0)] u_k'''(\rho_{m,k}), \end{aligned} \quad (90)$$

$$\partial_t \rho_{m,k} = -(d-4)\rho_{m,k} + 2v_d l_2^{(d)}(0), \quad (91)$$

which results in the nontrivial fixed point $\rho_{m*} = 2v_d l_2^{(d)}(0)/(d-4)$, $\lambda_* = (6-d)/(4v_d l_3^{(d)}(0))$. This fixed point is once unstable (and it remains so when considering the additional directions associated with the 2-replica potential, see above) and is characterized by critical exponents satisfying the dimension reduction property, *e.g.*, $\nu = 1/(d-4)$ to be compared to $\nu = 1/(d-2)$ for the pure model.

Note that the above perturbative expressions are recovered from the truncated NP-FRG equations even with an additional approximation using a field expansion around the minimum of the 1-replica potential.

B. Recovering the perturbative FRG near $D = 4$

A strong property of the minimal nonperturbative truncation described above is that it also reduces, in the appropriate limit and for the RFO($N > 1$)M, to the perturbative FRG equations at first order in $\epsilon = d-4$ derived by Fisher.²³ The latter are obtained from a low-disorder loop expansion of the nonlinear sigma model associated with the RFO(N)M. It is therefore quite remarkable that our formalism in which no hard constraint is enforced leads to the proper result within the minimal approximation scheme.

For the RFO(N)M with $N > 1$, $d = 4$ is the lower critical dimension for ferromagnetism. (We mean here long-range ferromagnetic order with a nonzero order parameter, the case of quasi-long range order will be discussed later on.) As a result, the critical point and the associated fixed point occur near $d = 4$ for a value of ρ_m that diverges as $1/\epsilon$ with $\epsilon = d-4$. As in the case of the pure $O(N)$ model near $d = 2$,³⁷ one can therefore organize a systematic expansion in powers of $1/\rho_m$.

At the minimum of the 1-replica potential ($\rho = \rho_m$), the transverse mass, associated with the Goldstone modes, is zero whereas the longitudinal mass is very large and scales as ρ_m (anticipating that $u''(\rho_m)$ does not vanish). One can then use the asymptotic properties of the threshold functions for large arguments,

$$l_n^{(d)}(w \rightarrow \infty) \sim w^{-(n+1)}, \quad l_{n_1, n_2}^{(d)}(w \rightarrow \infty, 0) \sim w^{-(n_1+1)}, \quad (92)$$

$$m_{n_1, n_2}^{(d)}(w \rightarrow \infty, 0) \sim w^{-n_1}, \quad (93)$$

which encodes the decoupling of the massive mode.

In addition, we assume that as $\rho_m \rightarrow \infty$, $\delta_{L,T}(\rho_m)$ stay finite (recall that actually, $\delta_T(\rho_m) = 1$) and that their derivatives, $\delta'_{L,T}(\rho_m)$, etc, go to zero at least as fast as $1/\rho_m$; on the other hand, ρ_m is a singular point for $u(\rho)$ (the location of its minimum), so that even when we expect that $u''(\rho_m)$, $u'''(\rho_m)$, etc, stay of $O(1)$. The consistency of these assumptions is easily checked *a posteriori*. Inserting the above results and assumptions in

Eqs. (79) gives

$$\eta_k \simeq \frac{8v_d}{d\rho_{m,k}}, \quad (94)$$

which shows that η is of order $1/\rho_m$.

Deriving once the flow equation for the 1-replica potential $u_k(\rho)$ leads to

$$\begin{aligned} \partial_t u'_k(\rho) = & -(2 - \eta_k)u'_k(\rho) + (\epsilon + \bar{\eta}_k)\rho u''_k(\rho) \\ & - 2v_d(N-1)l_2^{(d)}(u'_k(\rho))\delta_{k,T}(\rho), \end{aligned} \quad (95)$$

from which one obtains the flow equation for the running order parameter $\rho_{m,k}$:

$$\partial_t \rho_{m,k} = -(\epsilon + \bar{\eta}_k)\rho_{m,k} + 2v_d(N-1)l_2^{(d)}(0), \quad (96)$$

where $\epsilon = d-4$. (Note that we have again omitted the subscript k in the right-hand sides and dropped the subdominant terms involving the renormalized temperature T_k .) The last equation shows that the fixed point value of $\rho_{m,k}$ satisfies, as anticipated, $\rho_{m*} = O(1/\epsilon)$, which results in $\eta, \bar{\eta} = O(\epsilon)$.

One can now apply a similar treatment to the flow equation for the 2-replica potential evaluated for $\rho_1 = \rho_2 = \rho_{m,k}$. For convenience, we introduce the function

$$R_k(z) = \frac{v_k(\rho_{m,k}, \rho_{m,k}, z)}{(2\rho_{m,k})^2} \quad (97)$$

which, due to Eq. (74) and the constraint $\delta_{k,T}(\rho_{m,k}) = 1$, satisfies $R'_k(z=1) = 1/(2\rho_{m,k})$.⁷⁷ The flow equation for $R_k(z)$ can be expressed as

$$\begin{aligned} \partial_t R_k(z) = & \frac{1}{(2\rho_{m,k})^2} \partial_t v_k(\rho, \rho, z)|_{\rho=\rho_{m,k}} + \\ & \partial_t \rho_{m,k} \partial_\rho \left[\frac{v_k(\rho, \rho, z)}{(2\rho)^2} \right] \Big|_{\rho=\rho_{m,k}}, \end{aligned} \quad (98)$$

which with the help of Eq. (96) finally leads to

$$\begin{aligned} \partial_t R_k(z) \simeq & (\epsilon + 2\eta_k)R(z) - 2v_d l_2^{(d)}(0) \left\{ (N-1) \left[\right. \right. \\ & 4R_k(z)R'_k(1) + R'_k(z)(R'_k(z) - 2zR'_k(1)) \left. \right] + \\ & (1-z^2) \left[-R'_k(z)^2 + 2(R'_k(1) - zR'_k(z))R''_k(z) \right. \\ & \left. \left. + (1-z^2)R''_k(z)^2 \right] \right\}. \end{aligned} \quad (99)$$

To dominant order in ϵ , one can set $d = 4$ in v_d and $l_2^{(d)}(0)$ in all equations and in v_d/d in Eq. (94). Using the property of the threshold function $l_2^{(d=4)}(0) = 1 + O(\eta)$ and discarding subdominant terms, one finally arrives at

$$\eta_k = 4v_d R'_k(1), \quad \bar{\eta}_k = -\epsilon + 4(N-1)v_d R'_k(1), \quad (100)$$

$$\begin{aligned}
\partial_t R_k(z) = & \epsilon R_k(z) - 2v_4 \left\{ 4(N-2)R'_k(1)R_k(z) + \right. \\
& (N-1) \left[R'_k(z) - 2zR'_k(1) \right] R'_k(z) \\
& + (1-z^2) \left[-R'_k(z)^2 + 2(R'_k(1) - \right. \\
& \left. \left. zR'_k(z))R''_k(z) + (1-z^2)R''_k(z)^2 \right] \right\},
\end{aligned} \tag{101}$$

where $v_4^{-1} = 32\pi^2$ and $R_k(z)$ is of order ϵ near its fixed point. The above equations coincide with the one-loop perturbative FRG equations derived by Fisher.²³ Note that this result is independent of the choice of the infrared cut-off function $\widehat{R}_k(q^2)$: indeed, one easily checks that not only $l_2^{(4)}(0) = 1 + O(\eta)$, but also $\lim_{w \rightarrow \infty} [m_{2,3}^{(4)}(w, 0)w^2] = 1 + O(\eta)$, irrespective of the regulator.

Finally, we note that setting $N = 2$ and introducing the variable $\phi = \cos^{-1}(z)$ in Eq. (101) leads to

$$-\partial_t R_k(\phi) = \epsilon R_k(\phi) - 2v_4 [R''_k(\phi) - 2R''_k(0)] R'_k(\phi), \tag{102}$$

which, after use of Eq. (100) for η_k and $\bar{\eta}_k$, coincides with the 1-loop perturbative FRG equation for a disordered periodic elastic system with a one-component displacement field: compare for instance with Eq. (81), in which one should set $\zeta = 0$ due to the periodicity.⁷² (Be careful, however, that η_k and $\bar{\eta}_k$ denote different sets of exponents in the formalism leading to Eq. (81) and in the present one.)⁷⁸

VI. CONCLUDING REMARKS

In this work, described in the present paper and in the following one,³² we have developed a theoretical approach which is able to describe the long-distance physics, criticality, phase ordering or “quasi”-ordering, of systems in the presence of quenched disorder, in particular random field models for which standard perturbation theory is known to fail.

The approach is based on an exact renormalization group equation for the effective average action (the generating functional of $1 - PI$ vertices) and on a nonperturbative truncation scheme. This nonperturbative RG formalism has recently been applied with success to a variety of systems. The key point in the present problem is to provide a proper account of the renormalized distribution of the quenched disorder, and we have shown that this can be conveniently done through a cumulant expansion and the use of a replica method in which the permutational symmetry among replicas is explicitly broken.

We have stressed that any relevant treatment of random field models and related disordered systems must include the second cumulant of the renormalized disorder, *i.e.*, at least a function of two (replica) field argu-

ments. Accordingly, we have proposed a nonperturbative approximation scheme. Within this scheme, the minimal truncation for the RFO(N)M already reproduces the leading results of perturbative RG analyses near the upper critical dimension, $d_{uc} = 6$ and when the number of components N becomes infinite. More importantly, it gives back the perturbative FRG equations near the lower critical dimension for ferromagnetism when $N > 1$, $d = 4$.

One of the main advantages of the present approach, which will be illustrated in the following paper, is that it provides a unified framework to describe models in any spatial dimension d and for any number N of field components. As such, it guarantees a consistent interpolation of all known results in the whole (N, d) plane, in addition to allowing the study of genuine nonperturbative phenomena.

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APPENDIX A: EXPANSION IN SEVERAL COUPLING CONSTANTS NEAR $d = 6$

Near the upper critical dimension $d_{uc} = 6$, the flow equations for the RFIM derived within the minimal nonperturbative truncation, Eqs. (61, 62, 66, 69), can be expanded in several φ^4 coupling constants, in order to make the connection with recent one-loop studies of the RFIM^{69,70,71}. On top of the 1-replica φ^4 coupling constant already used in section V-A, $\lambda_k = u_k^{(4)}(\varphi_{m,k})$, we introduce two additional coupling constants obtained from the 2-replica potential,

$$u_{k,2} = -\frac{1}{2} (\partial_{\varphi_1}^2 + \partial_{\varphi_2}^2) \partial_{\varphi_1} \partial_{\varphi_2} v_k(\varphi_1, \varphi_2)|_{\varphi_1=\varphi_2=\varphi_{m,k}}, \tag{A1}$$

$$u_{k,3} = -\partial_{\varphi_1}^2 \partial_{\varphi_2}^2 v_k(\varphi_1, \varphi_2)|_{\varphi_1=\varphi_2=\varphi_{m,k}}, \tag{A2}$$

which amounts to consider a 2-replica potential of the form

$$\begin{aligned}
v_k(\varphi_1, \varphi_2) = & \varphi_1 \varphi_2 \left[\delta_{m,k} - \frac{u_{k,2}}{6} (\varphi_1^2 + \varphi_2^2 - 6\varphi_{m,k}^2) - \right. \\
& \left. \frac{u_{k,3}}{4} (\varphi_1 \varphi_2 - 4\varphi_{m,k}^2) + \dots \right],
\end{aligned} \tag{A3}$$

where the dots denote higher-order terms in the field expansion around the minimum of the 1-replica potential and, as before, $\delta_{m,k} = \partial_{\varphi_1} \partial_{\varphi_2} v_k(\varphi_1, \varphi_2)|_{\varphi_{m,k}} \equiv 1$. The present description is thus very similar to that used in Refs. [69,70], except that we do not include 3- and 4-replica terms. However, the issues raised by Brezin and De Dominicis^{69,70} can already be addressed by considering the 2-replica term.

By expanding the flow equations for the 1- and 2-replica potentials in powers of the coupling constants,

which we generically denote $u_{k,\alpha}$ with $u_{k,1} = \lambda_k$, one finds that $\eta = O(u^2)$ and that up to a $O(u^3)$,

$$\partial_t u_{k,1} = (d-6)u_{k,1} + 2v_d \left\{ 6l_3^{(d)}(0)u_{k,1}^2 + 12l_2^{(d)}(0)u_{k,1} \times (u_{k,2} + u_{k,3}) + 3T_k l_2^{(d)}(0)u_{k,1}^2 \right\}, \quad (\text{A4})$$

$$\partial_t u_{k,2} = (d-4)u_{k,2} + 2v_d \left\{ 6l_3^{(d)}(0)u_{k,1}(u_{k,2} + u_{k,3}) + 6l_2^{(d)}(0)u_{k,2}(u_{k,2} + u_{k,3}) + 3T_k l_2^{(d)}(0)u_{k,1}u_{k,2} \right\}, \quad (\text{A5})$$

$$\partial_t u_{k,3} = (d-4)u_{k,3} + 2v_d \left\{ l_4^{(d)}(0)u_{k,1}^2 + 4l_3^{(d)}(0)u_{k,1} \times (u_{k,2} + u_{k,3}) + 2l_2^{(d)}(0) [(u_{k,2} + u_{k,3})^2 + 3u_{k,3}^2] + 2T_k l_2^{(d)}(0)u_{k,1}u_{k,3} \right\}, \quad (\text{A6})$$

$$\partial_t \left(\frac{\lambda_k \rho_{m,k}}{3} \right) = -2 \left(\frac{\lambda_k \rho_{m,k}}{3} \right) + 2v_d \left\{ l_2^{(d)}(0)u_{k,1} + 2l_1^{(d)}(0)(u_{k,2} + u_{k,3}) + T_k l_1^{(d)}(0)u_{k,1} + \left(\frac{\lambda_k \rho_{m,k}}{3} \right) \times \left[2l_3^{(d)}(0)u_{k,1} + 8l_2^{(d)}(0)(u_{k,2} + u_{k,3}) + T_k l_2^{(d)}(0)u_{k,1} \right] \right\}, \quad (\text{A7})$$

with, we recall, $\rho_{m,k} = \varphi_{m,k}^2/2$. In addition, by using $\partial_t T_k = (2 + \eta_k - \bar{\eta}_k)T_k$ and the equation for $2\eta_k - \bar{\eta}_k$, Eq. (69), one obtains to a $O(u^2)$:

$$\partial_t T_k = 2T_k + 4v_d T_k \left\{ 2l_2^{(d)}(0)(u_{k,2} + u_{k,3}) + 6l_1^{(d)}(0) \times \frac{(u_{k,2} + u_{k,3})^2}{u_{k,1}} + T_k l_1^{(d)}(0)(2u_{k,2} + 3u_{k,3}) \right\}. \quad (\text{A8})$$

It can now be checked that the above equations coincide with those derived by Mukaida and Sakamoto⁷¹ with the introduction of new running coupling constants: $g_0 = T_k \delta_k(0)^{-1}$, $g_1 = u_{k,1} \delta_k(0)$, $g_2 = u_{k,2}$, $g_3 = u_{k,3}$, with $\delta_k(0) \equiv \delta_k(\varphi_1 = 0, \varphi_2 = 0) = 1 + 2(u_{k,2} + u_{k,3})\rho_{m,k}$ (and, of course, the 3- and 4-replica contributions missing). As these authors, we therefore obtain that the dimensional reduction fixed point is once unstable at first order in $\epsilon = 6 - d$.

On the other hand, the analysis performed by Brezin and De Dominicis^{69,70} requires to introduce different scaling dimensions, corresponding to new coupling constants, $\hat{g}_0 = g_0$, $\hat{g}_1 = g_1$, $\hat{g}_{2,3} = g_{2,3}g_0^{-1}$. The beta functions we now obtain for \hat{g}_0 , \hat{g}_1 and \hat{g}_2 coincide with those of Ref. [69,70], but that for \hat{g}_3 is ill defined as it contains a term that blows up as $k \rightarrow 0$. The scaling dimensions suggested by Brezin and De Dominicis are thus not compatible with our approach.

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- ⁷⁴ An alternative to the description of the random functional by its moments or cumulants would be to directly consider its probability distribution. Unfortunately, this turns out to be extremely involved in the context of the present exact RG approach and for the models under consideration. For the $3-d$ hard-spin lattice model a numerical study of this kind has been carried out within a real-space RG description in the Migdal-Kadanoff approximation.⁷³ For simpler cases, this alternative approach has been followed: see for instance Ref. [19].
- ⁷⁵ Note that the anomalous dimension η_k defined from the transverse (Goldstone) propagator in the RFO(N)M as in Eq. (79) and the exponent $\bar{\eta}_k$ defined from the transverse disorder strength as in Eq. (80) do not reduce to their RFIM counterparts given in Eqs. (66) and (69) when $N = 1$, since the latter are obtained from longitudinal quantities. This results from the nonuniqueness of the prescription used for defining renormalized quantities in a truncated RG approach. This is already present in the pure $O(N)$ model³⁶ and could be partly resolved by defining the anomalous dimension in the $O(N)$ model from an expression involving some arithmetic mean of transverse and longitudinal components (and similarly for the disorder strength). In practice, one often chooses to keep for the Ising model the $O(N)$ expression with $N = 1$. The ambiguity is more consistently resolved at the full first order of the derivative expansion⁶².
- ⁷⁶ The same property does not hold for the RAO(N)M with *e.g.* an initial condition $v_\Lambda(\rho_1, \rho_2, z) \propto \rho_1 \rho_2 z^2$, corresponding to $\delta_{\Lambda,T} \propto \sqrt{\rho_1 \rho_2} z$. Actually, it is known from our previous study near $d = 4$ that dimensional reduction is broken in this case: see Ref. [31].
- ⁷⁷ To make a more direct contact with the notations of Ref. [23], the function $R(z)$ used here differs by a factor $1/2$ from that used in our preceding Ref. [39].
- ⁷⁸ As will be discussed in more detail in the companion paper,³² the flow equations for the random elastic models are now considered for d close to, but less than, 4.